Rational maps represented by both rabbit and aeroplane matings

Thesis submitted in accordance with the requirements of
the University of Liverpool for the degree of Doctor in Philosophy
by

Freddie Rupert Exall

May 2011

Acknowledgements

I would like to express my deepest gratitude to Mary for guiding me through this work. Mary patiently pointed me in the right direction and offered endless encouragement without which this thesis would not exist. Thank you.

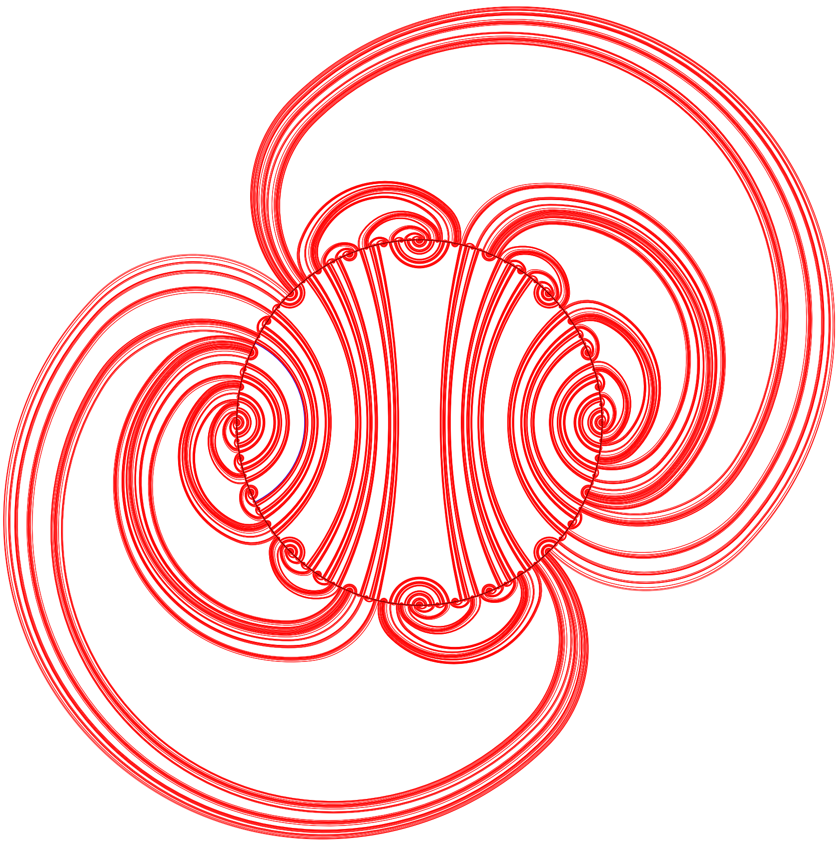

Figure 1: The Christmas example - μ_p joining $\frac{2035}{8191} - \frac{2052}{8191}$.

Rational maps represented by both rabbit and aeroplane matings

Freddie R. Exall

May 2011

Abstract

Understanding parameter spaces of rational maps is an active area of complex dynamics. There is a region of a particular parameter space of rational maps which contains all possible matings with the rabbit polynomial in a well understood manner. In an effort to further understand the which hyperbolic components of the parameter space correspond to matings with the aeroplane we relate the family of matings with the aeroplane to the family of matings with the rabbit.

We present an algorithm, described in chapter 3, which calculates the mating with the rabbit which is Thurston equivalent to a given post-critically finite mating with the aeroplane. Chapter 4 gives a result describing which matings with the rabbit are Thurston equivalent to some mating with the aeroplane. Chapter 5 studies the algorithm in more detail, giving results bounding the number of steps required for the algorithm to produce a result.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

This work aims to further understanding of a parameter space of degree two rational maps. There are two families of maps for which the understanding of one is much more complete than that of the other. By relating these families of maps it is hoped that evidence for the structure of the less well understood family will be found.

Any map $f : X \to X$ may be considered as a dynamical system. For any $x \in X$ the map f 'moves' x to $f(x)$. Further applications of f create an itinerary of x under f. The behaviour of this itinerary, when X has a complex structure, is the subject of study in complex dynamics. To simplify this study it is often useful to categorise maps by certain behaviour. For this a *parameter space* may be employed.

A parameter space catalogues a group of objects by an indexing parameter. Perhaps the most famous parameter space in mathematics, the Mandelbrot set $\mathcal{M}$, catalogues the degree two polynomials of the form $f_c : z \mapsto z^2 + c$ using the parameter $c \in \mathbb{C}$. Any degree two polynomial can be conjugated by an affine transformation to a polynomial of the form $z \mapsto z^2 + c$ and so understanding the Mandelbrot set gives an understanding of all degree two polynomials. The point $c \in \mathbb{C}$ lies in $\mathcal{M}$ if and only if the map $f_c : z \mapsto z^2 + c$ is such that $|f_c^{\circ n}(0)|$ is bounded for all n. Here

$$f^{\circ n} = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}.$$

This parameter space has been studied since the resurgence of complex dynamics around 1980. Much of the initial work was completed by Douady and Hubbard. With the work of many others the set $\mathcal{M}$ is now mostly well understood. In the seminal paper [DH1] *parameter rays* are discussed which rephrase the structure of $\mathcal{M}$ in terms of pairs of arguments in $[0,1]/\sim$ (where $0 \sim 1$). Thurston re-interpreted this work to develop the theory of *laminations* in [TH]. These laminations are central to this piece of work and are used throughout.

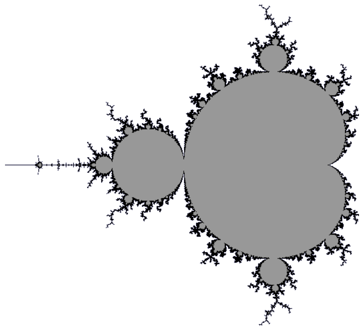

Figure 1.1: The Mandelbrot set, $\mathcal{M}$. Points in $\mathcal{M}$ are shaded.

A map is called *hyperbolic* if the forward orbit of each of its critical points tends to a periodic orbit. In a parameter space the maximal open sets of hyperbolic maps are called *hyperbolic components*. One reason why the concept of hyperbolicity is so powerful is that hyperbolic maps which are in the same hyperbolic component share much of the same structure; they are topological conjugate on some significant set. As all maps in any hyperbolic component have a similar structure a dynamicist need only understand a single map from each component.

The *centre* of a hyperbolic component is the map for which the post-critical set, that is the union of the forward orbits of all critical points, is finite. A hyperbolic component has a centre if and only if the maps contained in the component only have simply connected Fatou components. In [R1] it is shown that all Fatou components are simply connected for degree two rational maps which do not lie in the hyperbolic component containing f_c, for all $c \notin \mathcal{M}$.

Rational maps may also be considered in the study of dynamical systems and often exhibit very complex behaviour. While any such rational map can be considered, this thesis will consider only hyperbolic rational maps. While degree two polynomials have a well understood parameter space no such space exists for the superset of degree two rational maps. Methods of constructing rational maps from polynomials offer one way of relating a parameter space of rational maps to $\mathcal{M}$.

First discussed by Douady and Hubbard in 1985 (see [DH2]) a *mating* is a combination of two polynomials. The process of mating combines the dynamics of two polynomial maps to create a map with richer dynamics. Thurston developed a criterion which provides a mechanism to determine whether or not a mating between critically finite polynomials is equivalent, in some homotopy like way, to a rational map. Investigating further, Tan and Rees were able to formulate a simple statement which describes precisely which matings, between pairs of critically finite polynomials, are *admissible*, that is, Thurston equivalent to rational maps: a mating between polynomials f_a and f_b is admissible if, and only if, f_a and f_b do not lie in conjugate limbs of the Mandelbrot set.

In previous work, Rees labels the maps which have two distinct periodic critical orbits *type IV*. These type IV maps are the centres of the hyperbolic components, in any parameter space of degree two rational maps, which contain maps with two disjoint periodic orbits of Fatou components. Much of the theory presented in this document aims to describe these type IV maps with the understanding that this gives information about the containing hyperbolic components.

The work contained in this document is concerned with a particular parameter space. Identifying each point $a \in \mathbb{C} \setminus 0$ with the map

$$h_a(z) = \frac{(z-a)(z-1)}{z^2}$$

gives a parameter space of degree two rational maps. This parameter space, which we will call V_3, contains rational maps of which one critical point, $c_1 = 0$, has orbit $\{0, \infty, 1\}$. Also, V_3 does not intersect the hyperbolic component in the space of all degree two rational maps containing f_c for $c \notin \mathcal{M}$ and so all hyperbolic components of V_3 contain a centre. Figure 1.2 is a somewhat accurate image of V_3.

The large, bounded hyperbolic components in the centre of figure 1.2 contain the points 1 and -1 and meet at three points, one of which is 0. These hyperbolic components bound two regions; label the bounded region contained in the upper half plane b_ω and the conjugate region $b_{\overline{\omega}}$ (see figure 1.3).

All matings in V_3 lie in hyperbolic components which contain type IV

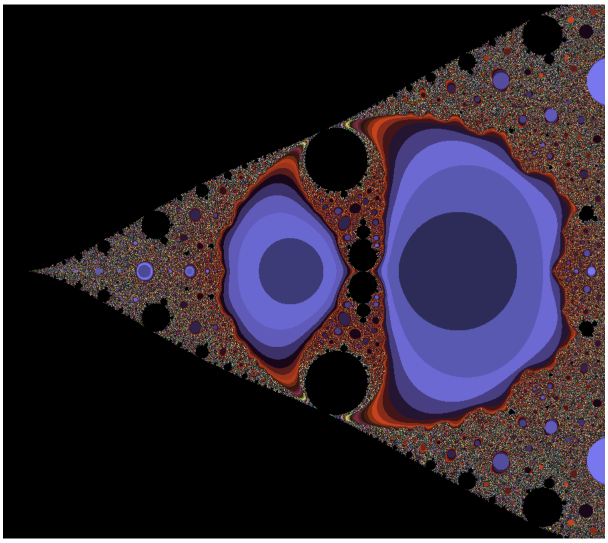

Figure 1.2: The parameter space V_3.

maps. So, understanding the type IV maps gives an understanding of all matings in V_3. This thesis explores these type IV maps by considering matings between centres of components of $\mathcal{M}$.

It is known that all type IV maps in b_ω are matings with Douady's rabbit polynomial (f_ω where $(\omega^2+\omega)^2+\omega = 0$ and $\Im(\omega) > 0$), as can be seen in section 2.8 of [R6]. The embedding of matings with the rabbit polynomial forms a copy of $\mathcal{M}$, $\mathcal{M}_\omega$, which has the anti-rabbit limb removed. The outside of $\mathcal{M}_\omega$ is identified with a copy the Julia set of $f_{\overline{\omega}}$, which has had the Fatou component containing the critical value removed (some of this structure can be seen in figure 1.4).

Clearly, in light of this embedding of the matings with the rabbit polynomial as an almost complete copy of $\mathcal{M}$ in V_3 there is hope that other families of matings may embed in a similarly well understood manner. Given that the critical point c_1 of $h_a \in V_3$ is of period three for all a, the only families of matings which will be present in V_3 are matings with the rabbit, the anti-rabbit, and the aeroplane polynomial (f_α where $(\alpha^2+\alpha)^2+\alpha = 0$, $\Im(\alpha) = 0$ and $\Re(\alpha) < 0$). Matings with the rabbit are contained in b_ω and similarly matings with the anti-rabbit are contained in $b_{\overline{\omega}}$. Hence, it remains to understand how the family of matings with the aeroplane polynomial is embedded in V_3 (that

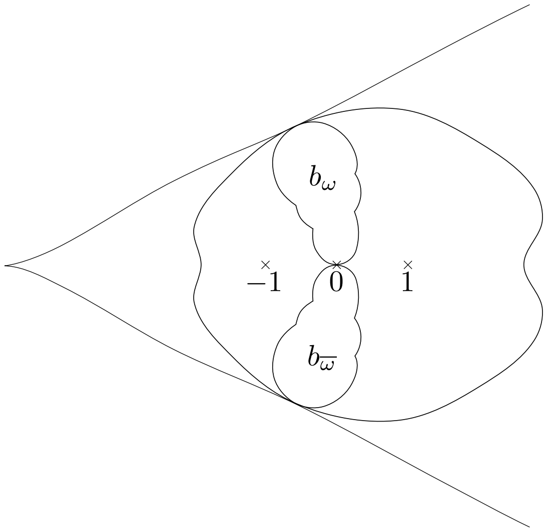

Figure 1.3: A labelling of V_3

Figure 1.4: Detail of b_ω

is, to understand the subset of V_3 consisting of maps with two disjoint critical orbits, one of which has period three). Work by Rees ([R4], [R5] and [R6]) and unpublished work by Adam Epstein exists in this area.

One complication which stands in the way of understanding the type IV components of V_3 is that a rational map may be Thurston equivalent to more than one mating.

In [W], Wittner found that the aeroplane polynomial mated with the rabbit polynomial is Thurston equivalent to the mating of the same two polynomials in the reverse order. This shows that the set of matings with the aeroplane intersects b_ω. Work presented in this thesis investigates the structure of this intersection.

Take X to be the set of type IV maps which are equivalent to matings of the aeroplane polynomial with something from the rabbit limb. It can be shown that $X \subsetneq b_\omega$. The focus of this work is on the embedding of X as a subset of $\mathcal{M}_\omega$. The main tool used in this investigation is an algorithm which, given a $h \in X$, can calculate the equivalent mating with the rabbit. Working with Thurston's lamination model of the Julia set of a polynomial, with the associated maps (see section 2.2.1), symbolic dynamics is harnessed to create a 'tableau' from which the output is calculated.

One might hope that X embeds into $\mathcal{M}_\omega$ in some simply describable manner. It is shown that all maps in X correspond to matings of some polynomial in the aeroplane limb with the rabbit, which neatly parallels results in [W] (discussed briefly at the beginning of chapter 4). The equivalences calculated by the algorithm, presented in appendix B, do not immediately indicate any further pattern, however. There is further work to be done to determine whether the embedding of the set X is connected as the equivalences from the truncated anti-rabbit Julia set on the exterior of $\mathcal{M}_\omega$ have not been considered.

Other results presented here concern the algorithm itself. An upper bound is sought for the number of iterations required in the algorithm for a definitive answer to be produced. An upper bound for the general case is found but it is not sharp. Experimental data suggests that a much sharper bound is possible. For particular families of examples a sharp bound can and has been found.

1.0.1 Document Structure

Chapter 2 introduces the necessary objects to discuss the theory presented in later chapters. A brief review of Thurston's laminations is presented in section 2.2. The concept of mating is introduced in section 2.3. Thurston equivalence is defined and particular cases of equivalent matings are also discussed.

Chapter 3 introduces the algorithm mentioned above. The supporting theory is introduced before the steps of the algorithm are described in detail in section 3.1. Worked examples are provided in section 3.2.

The algorithm produces a map from a mating with the aeroplane to a mating with the rabbit. Chapter 4 investigates the image of this map. A result showing that the map has an image consisting of polynomials in the aeroplane limb mated with the rabbit is given.

Chapter 5 is concerned with the number of steps required for the algorithm to reach a result. Three families of examples for which a sharp bound is found are presented in section 5.1 before the general case is considered in 5.2.

Appendix A briefly discusses the programs which were created to support the work in this document. The source code is not reproduced in this appendix but it is the intention of the author to submit the related source code with an electronic copy of this thesis to the University of Liverpool library in the event of graduation, where it will be readily available.

Appendix B lists equivalent matings as computed by the implementation of the algorithm described in appendix A.

Chapter 2

A background in laminations and mating

This chapter serves to introduce the basic objects used in this document. While Thurston's laminations, in section 2.2, are the first of these a number of more standard definitions are required beforehand. The definitions presented here will be familiar to many.

Definition 2.0.1 *A polynomial function,*

$$f(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0,$$

is said to be degree two if and only if $k = 2$.

Definition 2.0.2 *A rational function is a function*

$$R(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials. A degree two rational function is such that the maximum degree of $P(z)$ and $Q(z)$ is two.

Definition 2.0.3 *A* normal family *from $S \to \overline{\mathbb{C}}$, where $S \subset \overline{\mathbb{C}}$, is a family of maps $\{f_i\}$ such that any infinite sequence of functions in $\{f_i\}$ contains a subsequence which converges locally uniformly on S.*

Definition 2.0.4 *The* Fatou set *of a function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the set $F(f)$ where $f^{\circ n}|_{F(f)}$ forms a normal family (here $f^{\circ n}$ means the map f applied n times).*

Definition 2.0.5 *The* Julia set *of a function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $J(f)$, is the complement of the Fatou set.*

While containing no further information from the Julia set, the filled Julia set, from definition 2.0.6, will be of particular interest in section 2.2.1. Similarly, definition 2.0.7 is required for the understanding of Thurston equivalence introduced in the same section.

Definition 2.0.6 *The* filled *Julia set of a polynomial function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, denoted $K(f)$, is simply the complement of the Fatou component of f containing infinity.*

Definition 2.0.7 *Define the* post-critical set *of a function, $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, to be the set*

$$X(f) = \{f^{\circ k}(z) : f'(z) = 0, k \in \mathbb{N}\}$$

In this document we use degree two hyperbolic polynomials as fundamental objects.

Definition 2.0.8 *A* hyperbolic *function, $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, is such that the closure of the post critical set is disjoint from the Julia set.*

Definition 2.0.9 *In a parameter space, a maximal connected component of hyperbolic maps is called a* hyperbolic component*.*

In fact, the functions considered in the rest of this document will be even more precise than simply hyperbolic. They are the so-called "centres" of hyperbolic components of the Mandelbrot set.

Definition 2.0.10 *The* centre *of a hyperbolic component is the hyperbolic map in the component whose critical points are periodic.*

Throughout the remainder of this document we will be using centres of hyperbolic components of the Mandelbrot set, that is, degree two polynomials with periodic critical point, to construct degree two rational maps.

Definition 2.0.11 *Consider a hyperbolic polynomial map f with critical point c_f. Up to equivalence we can assume that there are open neighbourhoods U_1*

and U_2 of c_f which are topological discs whose closures are closed topological discs with $U_1 \subset \overline{U_1} \subset U_2$. Further, we may assume that U_1 is a component of $f^{-n}(U_2)$, that U_2 contains no point of the postcritical set apart from c_f, and that U_1 is mapped with degree two onto U_2 by f^n.

Take the map g to be a hyperbolic quadratic polynomial up to equivalence (for example, the corresponding lamination map, see section 2.2.1). We may assume that ∞ is the fixed critical point and c_g is the other critical point possibly of higher period. Set V_1, V_2 to be bounded open topological discs whose closures are closed topological discs which contain the entire postcritical set, apart from ∞. Also, let $V_1 = g^{-1}(V_2)$ where $\overline{V_1} \subset V_2$. Then (by covering space theory) there is a homeomorphism $\psi : \overline{U_2} \to \overline{V_2}$ with $\psi \circ f^n = g \circ \psi$ on ∂U_1. Then $f \vdash g$, the tuning *of f by g, is defined by*

$$f \vdash g(z) = \begin{cases} f(z) & : z \notin U_1, \\ f^{\circ(1-n)} \circ \psi^{-1} \circ g \circ \psi(z) & : z \in U_1, \end{cases}$$

so that $(f \vdash g)^{\circ n}(z) = \psi^{-1} \circ g \circ \psi(z)$ for $z \in U_1$.

2.1 Thurston Equivalence and Semi-Conjugacy

We wish to introduce and discuss laminations, and their relation to polynomials. To do this we must first discuss an equivalence relation, which will be used extensively throughout the rest of this thesis. We then proceed to discuss semi-conjugacy; a concept which is closely related to this equivalence.

The equivalence used is *Thurston equivalence*, which is a homotopy relation with a number of further conditions. We provide both a more standard definition followed by a alternative form more suitable to our setting.

Definition 2.1.1 *Take two critically finite branched coverings, f_0, f_1, and two ordered finite sets $X_0 = \{x_{0,0}, x_{0,1}, x_{0,2}, \ldots, x_{0,k}\} \in \overline{\mathbb{C}}$ and $X_1 = \{x_{1,0}, x_{1,1}, x_{1,2}, \ldots, x_{1,k}\} \in \overline{\mathbb{C}}$ where $f_i(X_i) \subset X_i$. Then (f_0, X_0) and (f_1, X_1) are* Thurston equivalent *if there are homeomorphisms $\varphi_0 : (\overline{\mathbb{C}}, X_0) \to (\overline{\mathbb{C}}, X_1)$ preserving the numbering on X_0 and X_1 and $\varphi_1 : (\overline{\mathbb{C}}, f_0^{-1}(X_0)) \to (\overline{\mathbb{C}}, f_1^{-1}(X_1))$ such that*

$$f_1 \circ \varphi_1 = \varphi_0 \circ f_0$$

and

$$\varphi_1 \simeq \varphi_0 \text{ rel } X_0$$

that is, φ_0 and φ_1 are isotopic via an isotopy which is constant on X_0. We may write

$$(f_0, X_0) \simeq_{\varphi_0} (f_1, X_1).$$

While definition 2.1.1 is the more usual form for the definition we will instead be using definition 2.1.2 in this thesis.

Definition 2.1.2 *Let (f_0, X_0) and (f_1, X_1) be as in definition 2.1.1. Then, if there exists a homotopy $F_t(z) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, where F_t is a critically finite branched covering for all t, and an isotopy $\varphi_t(z) : X_0 \to \overline{\mathbb{C}}$ where*

$$\begin{aligned} F_0(z) &= f_0(z), \\ F_1(z) &= f_1(z), \\ \varphi_0(x_{0,j}) &= x_{0,j}, \\ \varphi_1(x_{0,j}) &= x_{1,j}, \\ F_t(\varphi_t(X_0)) &\subset \varphi_t(X_0), \end{aligned}$$

then (f_0, X_0) and (f_1, X_1) are Thurston equivalent.

Lemma 2.1.1 *The definitions, 2.1.1 and 2.1.2, of Thurston equivalence are equivalent.*

Proof: See [R2], section 1.4. ■

Given a Thurston equivalence $(f_0, X_0) \simeq (f_1, X_1)$ lemma 2.1.2 shows that it is possible to find an equivalence between the two maps on the pull-backs of the two sets. Another form of this lemma is presented in [R2].

Lemma 2.1.2 *If*

$$(f_0, X_0) \simeq_{\varphi_0} (f_1, X_1)$$

then recursively defining φ_n by

$$\begin{aligned} \varphi_{n-1} \circ f_0 = f_1 \circ \varphi_n \qquad & \textit{and} \\ \varphi_{n-1} \simeq \varphi_n \ \textit{rel}\ f_0^{-(n-1)}(X_0). & \end{aligned}$$

gives the Thurston equivalence

$$(f_0, f_0^{-n}(X_0)) \simeq_{\varphi_n} (f_1, f_1^{-n}(X_1)).$$

Proof: Both maps

$$f_0 : \overline{\mathbb{C}} \setminus f_0(X_0) \to \overline{\mathbb{C}} \setminus X_0$$

and

$$f_1 : \overline{\mathbb{C}} \setminus f_1(X_1) \to \overline{\mathbb{C}} \setminus X_1$$

are coverings, necessarily of the same degree. Hence, there exists a continuous map

$$(t, z) \mapsto \varphi_{t+1}(z) : [0, 1] \times (\overline{\mathbb{C}} \setminus f_0^{-1}(X_0)) \to \overline{\mathbb{C}} \setminus f_1^{-1}(X_1)$$

such that

$$\varphi_t \circ f_0 = f_1 \circ \varphi_{t+1}$$

(in particular, $\varphi_1 \circ f_0 = f_1 \circ \varphi_2$) and the two definitions of φ_1 (from φ_t with $t = 1$ and φ_{t+1} with $t = 0$) coincide. Since φ_1 is a homeomorphism from $\overline{\mathbb{C}} \setminus f_0(X_0)$ to $\overline{\mathbb{C}} \setminus f_1(X_1)$ it follows that φ_{t+1} is also a homeomorphism from $\overline{\mathbb{C}} \setminus f_0(X_0)$ to $\overline{\mathbb{C}} \setminus f_1(X_1)$ for all $t \in [0, 1]$. Since φ_1 is a homeomorphism of $\overline{\mathbb{C}}$, it follows that φ_{t+1} also extends to a homeomorphism of $\overline{\mathbb{C}}$ for all $t \in [0, 1]$. Hence φ_{t+1} is an isotopy between φ_1 and φ_2 for $t \in [0, 1]$ and

$$\varphi_{t+1}(f_0^{-1}(X_0)) = f_1^{-1}(X_1), \quad t \in [0, 1]$$

Similarly by induction we have

$$\varphi_n \circ f_0 = f_1 \circ \varphi_{n+1}$$

$$\varphi_n(f_0^{-n}(X_0)) = \varphi_{n+1}(f_0^{-n}(X_0)) = f_1^{-n}(X_1).$$

and there is an isotopy (which we could call φ_{t+n}) between φ_n and φ_n, which is constant on $f_0^{-n}(X_0)$. ■

The following proposition is taken directly from [R2]

Proposition 2.1.3 *Let f_1 be a critically finite rational map, and let f_0 be a critically finite branched covering with $(f_0, X(f_0)) \simeq_{\varphi_0} (f_1, X(f_1))$. Take the homeomorphisms φ_n to be defined as in lemma 2.1.2. Then the sequence φ_n extends to a continuous path of homeomorphisms φ_t ($t \in [0, \infty)$) such that $\varphi_\infty = \lim_{t\to\infty} \varphi_t$ exists as a uniform limit, and*

$$\varphi_\infty \circ f_0 = f_1 \circ \varphi_\infty.$$

2.1.1 Thurston's Theorem

We now work towards presenting theorem 2.1.4, which provides the motivation for this thesis (together with 2.3.1). Some definitions must be made before the statement, however.

Definition 2.1.3 *A* Levy cycle for (f, X) *is a finite set of loops* $\{\gamma_i : 1 \leq i \leq r\}$ *in* $\overline{\mathbb{C}} \setminus X$ *such that* γ_{i-1} *is isotopic in* $\overline{\mathbb{C}} \setminus X$ *to a component* γ'_{i-1} *of* $f^{-1}(\gamma_i)$, *and* $f|\gamma'_{i-1}$ *is a homeomorphism. Here, we write* $\gamma_0 = \gamma_r$ *and* $\gamma'_0 = \gamma'_r$ *if* $i = 1$

Definition 2.1.4 *A Levy cycle* $\{\gamma_i : 1 \leq i \leq r\}$ *is* degenerate *if in addition each* γ_i *bounds a disc* D_i *and* γ'_i *bounds a disc* D'_i *such that the isotopy of* γ'_i *to* γ_i *maps* D'_i *to* D_i.

Definition 2.1.5 *A* Thurston obstruction *for* f *is a formal sum, with real coefficients, of isotopy classes of disjoint simple non-trivial non-peripheral closed loops, in the complement of the post-critical set of* f, *which is an eigenvector under* f^* *with eigenvalue greater than or equal to* 1, *where* f^* *is defined in a natural way by taking inverse images under* f. *For* f *degree* 2, *all Thurston obstructions are generated by Levy cycles.*

Theorem 2.1.4 (Thurston's Theorem for degree 2) *Let* f *be an orientation-preserving degree two branched covering of the two-sphere for which both critical points are periodic. Let* $X(f)$ *be any finite set containing the critical orbits and with* $f(X(f)) \subset X(f)$ *Then* $(f, X(f))$ *is Thurston equivalent to a rational map if and only if there is no Levy cycle for* f. *Moreover, if it exists, then* f *is unique up to Möbius conjugacy preserving numbering of critical points.*

Theorem 2.1.5 is an adaption of the general statement of Thurston's Theorem to address polynomials.

Theorem 2.1.5 *Let* $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *is a critically finite branched covering of degree* d *such that one critical point is fixed and of multiplicity* $d-1$ *and let* $X(f)$ *be finite containing the post-critical set. Suppose that there are no degenerate Levy cycles for* $(f, X(f))$. *Then* $(f, X(f))$ *is Thurston equivalent to* $(g, X(g))$ *for a polynomial* g. *Moreover* $(g, X(g))$ *is unique up to Möbius conjugacy preserving the numbering of* $X(g)$.

Proof: If D is any topological disc disjoint from the fixed critical value v of f of multiplicity $d-1$, then $f^{-1}(D)$ is a union of such discs, whose intersection with $X(f)$ is $X(f) \cap f^{-1}(D)$. It follows that if there are no degenerate Levy cycles, then the minimum degree of f^n restricted to a component of $f^{-n}(D)$ which intersects $X(f)$ tends to infinity with n. It follows that there are no Thurston obstructions. ■

2.2 Laminations

This section introduces *laminations*, a standardised form which we will later use to represent the Julia set of a function. Based on the work in [TH] these simple objects allow a convenient setting for employing symbolic dynamics. This then provides a notation which can be used to express the method for finding equivalent matings, to be discussed later.

Definition 2.2.1 *A* leaf *on the unit disc is a path connecting two points on the unit circle. A* lamination*, L, is a union of leaves and the unit circle which satisfies*

- *leaves do not cross (although they may share endpoints) and*
- *L is a closed set.*

A given function, s, which preserves the unit circle, then acts on a lamination by moving the leaves. This action is defined by

$$s(\ell_{p,q}) = \ell_{s(p),s(q)},$$

where $\ell_{a,b}$ is the leaf with a and b as endpoints. A gap *of a lamination (the closure of the complement of L in $\mathbb{D}$) is mapped according to the image of its boundary leaves. A lamination must also be* invariant *under the action of s, that is*

- *for any leaf $\ell \in L, s(\ell) \in L$,*
- *for any leaf with endpoints p and q, each pre-image of p is joined to a pre-image of q by some leaf in L and*
- *a* gap *is sent to another gap, a leaf, or a point by s.*

Figures 2.1 to 2.4 are provided as examples of laminations.

Definition 2.2.2 *A lamination is* clean *if all leaves connected to a single point lie on the boundary of a common gap.*

The laminations that are used from section 2.2.1 are constructed so as to be clean. The notation for laminations used from here on is as such: the lamination with *minor leaf* μ_q is denoted L_q.

Definition 2.2.3 *The* minor leaf *of a lamination is the image of the longest leaves in the lamination. Equivalently, the minor leaf is the shortest leaf in its own forward image. The minor leaf with endpoints at p and q is denoted*

$$\mu_p \quad \text{or equivalently} \quad \mu_q.$$

A brief note on notation: when labelling points on the unit circle, such as endpoints of leaves, we take the rational coefficient q in the exponent $e^{2\pi iq}$ for simplicity. For example we label $1/\sqrt{2} + i/\sqrt{2}$ by $1/8$ and $-1/2 - i\sqrt{3}/2$ by $2/3$. This means $[0,1]/\sim$, where $0 \sim 1$, labels all points on S^1.

It is possible to generate a lamination from a minor leaf. To do this start with only the minor leaf on the unit disc, connecting two points of S^1. Pulling back the minor leaf, choosing the long pre-images, gives the major leaves. Continue to pull back leaves choosing pre-images of leaves so that no leaves are longer than the major leaves and no leaves cross the major leaves. To complete the lamination add in any limit leaves.

In the following chapters the distance between leaves is mentioned. Definition 2.2.4 outlines the distance function referred to.

Definition 2.2.4 *The distance between two points, $e^{2\pi i\alpha}$ and $e^{2\pi i\beta}$, on S^1 in the context of laminations is the minimum euclidean distance between α and β in $[0,1]/\sim$, where $0 \sim 1$. The distance between two leaves is simply the minimum distance between their endpoints.*

2.2.1 The relationship between Laminations and Julia Sets

The aim of this document is to explore the maps derived by combining two degree two hyperbolic polynomials by *mating* them (see 2.3). The algorithm

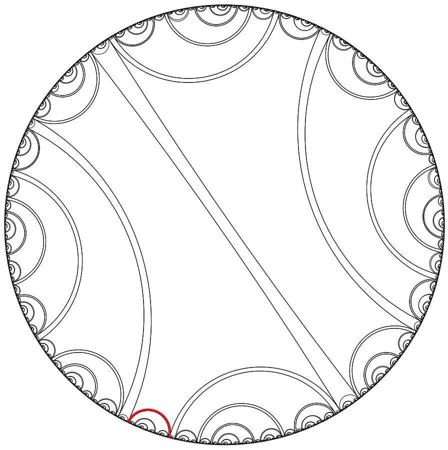

Figure 2.1: The lamination $L_{21/31}$ whose minor leaf has endpoints 21/31 and 22/31.

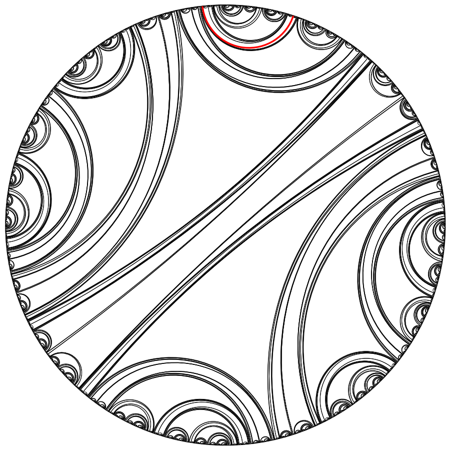

Figure 2.2: The lamination $L_{1/5}$ whose minor leaf has endpoints 1/5 and 4/15.

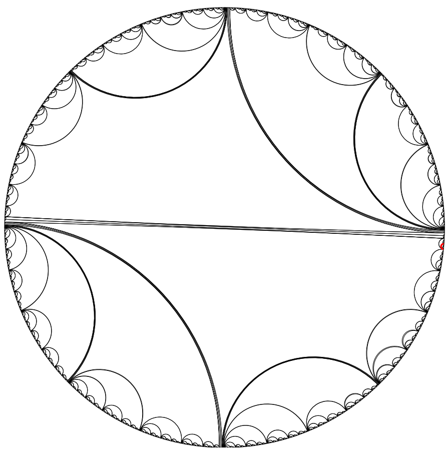

Figure 2.3: The lamination $L_{84/85}$ whose minor leaf has endpoints 84/85 and 251/255.

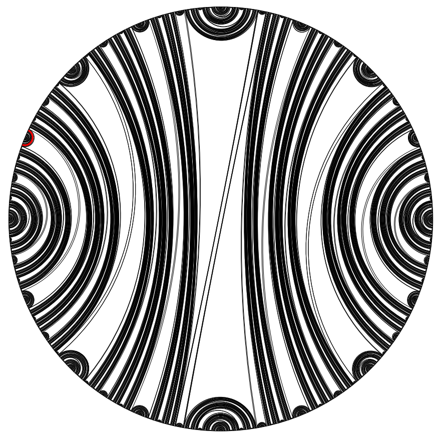

Figure 2.4: The lamination $L_{22/51}$ whose minor leaf has endpoints 22/51 and 113/255.

described in chapter 3 operates on laminations due to their susceptibility to symbolic dynamics. It is therefore necessary to understand the correspondence between laminations and Julia sets, where it exists.

Theorem 2.2.1 (Böttcher, 1904, [M]) *If* 0 *is a critical point of a map,* f*, of degree* d *and* 0 *is a superattracting fixed point then there exists a local holomorphic change of coordinate* φ *such that* $\varphi(0) = 0$ *and which conjugates* f *to the power map* $z \mapsto z^n$ *in some neighbourhood of* 0*. The map* φ *is unique up to composition with multiplication by a* $(d-1)^{th}$ *root of unity.*

Any polynomial on the plane may be extended to a polynomial on the sphere which sends infinity to itself. This makes infinity a superattracting fixed point of f. Böttcher's theorem can be used to normalise some neighbourhood of infinity to some neighbourhood of zero in this case. The following theorem is an extension of Böttcher's which serves to specify exactly what conditions are needed to have a map which can be represented by a lamination.

Theorem 2.2.2 ([M]) *If all the finite critical points of a map are contained within the filled Julia set,* K*, then* K *is connected, with its complement being conformally isomorphic to the exterior of the closed unit disc via the isomorphism* $\varphi : \mathbb{C} \setminus K \to \mathbb{C} \setminus \overline{\mathbb{D}}$*. Further to this,* f *conjugates to the map* $z \mapsto z^d$*, on the exterior of the closed disc, under* φ*.*

If there exists a finite critical point not contained in K *then* K *and, hence,* $J(f)$ *consist of uncountably many connected components.*

As the maps we're considering lie in the interior of the Mandelbrot set their filled Julia sets are connected, as required by theorem 2.2.2.

Let $\psi = \varphi^{-1}$. Given that the critical points, and their images, of the map f are bounded the map ψ is defined on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. The map ψ is not defined on S^1 necessarily. The following theorem illustrates when ψ can be extended to S^1.

Theorem 2.2.3 (Carathéodory, [M]) *A conformal isomorphism* $\psi : \mathbb{D} \to U \subset \overline{\mathbb{C}}$ *extends to a continuous map from the closed disc,* $\overline{\mathbb{D}}$*, to* $\overline{U}$ *if and only if the boundary,* ∂U*, is locally connected.*

We look only at hyperbolic maps, giving that our maps automatically satisfy the conditions of theorem 2.2.3 (connected Julia sets of hyperbolic maps are locally connected, see [M]). This gives that the inverse, ψ, to the Böttcher map can be extended to the boundary of the disc, for the maps we are considering, giving that the diagram

$$\begin{array}{ccc} \overline{\mathbb{C}} \setminus \mathrm{int}(K) & \xrightarrow{f} & \overline{\mathbb{C}} \setminus \mathrm{int}(K) \\ \psi \uparrow & & \uparrow \psi \\ \overline{\mathbb{C}} \setminus \mathbb{D} & \xrightarrow[z \mapsto z^d]{} & \overline{\mathbb{C}} \setminus \mathbb{D} \end{array}$$

commutes.

Definition 2.2.5 *The action of f can be taken to be $z \mapsto z^d$ on $\overline{\mathbb{C}} \setminus \mathbb{D}$, in light of theorem 2.2.3. This action extends onto S^1 and sends an equivalence class of points of S^1 to another. Hence, this action sends leaves to leaves on L_q. From this action a map $s_q : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ can be defined (up to homotopy with respect to S^1) by taking the action of $z \mapsto z^d$ on S^1, mapping leaves to leaves in accordance with its action on leaves' endpoints, and sending gaps to gaps in accordance with its action on a gaps boundary leaves.*

Due to the fact that s_q is only defined up to homotopy with respect to S^1 the leaves of a lamination may be any continuous path from one endpoint to another. Throughout this document, for reasons of clarity, leaves will be drawn as geodesics on the hyperbolic disc when drawn on the interior of S^1, and approximations of hyperbolic geodesics on the exterior. This is not to imply the use of any hyperbolic metric; indeed any metrics discussed will apply to the endpoints of leaves, on S^1, with the usual Euclidean metric on the interval. Also, for laminations on the interior of S^1 we will define s_q such that 0 is the critical point and for laminations on the exterior of S^1 we define s_q such that ∞ is the critical point.

We can construct a lamination, L_q, using the map ψ. Connect with leaves each subset of S^1 which has a single image point under ψ. This set consists of finitely many points. In this way we treat the leaves as the illustration of an equivalence: two points are equivalent if, and only if, they share an image point under ψ. Connecting each point in an equivalence class with a leaf would not necessarily yield a clean lamination. Instead, we connect points in the equivalence class with leaves only if they are adjacent, as points on the unit circle, in the class. This yields equivalence classes joined by a line or a polygon in a clean lamination we associate with the map f.

Lemma 2.2.4 *Using the notation of the previous paragraph, f is Thurston equivalent to s_q.*

Proof: To see this, note that from above we have that $f \circ \psi = \psi \circ s_q$ where $\psi : \{|z| \geq 1\} \to \{\overline{\mathbb{C}} \setminus K\} \cup J(f)$ is a homeomorphism on $\{|z| > 1\}$. There exist homeomorphisms ψ_i such that $lim_{i \to \infty} \psi_i = \psi$ as ψ is a semi-conjugacy. Take $\psi_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ to be a homeomorphism close to ψ. Then

$$\psi \circ s_q \circ \psi_n^{\circ -1} = f \circ \psi \circ \psi_n^{\circ -1}.$$

Now $f \circ \psi \circ \psi_n^{\circ -1}$ is close to $f \circ \psi_n \circ \psi_n^{\circ -1} = f$. From the fact that $\psi \circ s_q \circ \psi_n^{\circ -1}$ is also close to $\psi_n \circ s_q \circ \psi_n^{\circ -1}$ we see that f and $\psi_n \circ s_q \circ \psi_n^{\circ -1}$ are close, and can be made arbitrarily close by choice of n. As both f and $\psi_n \circ s_q \circ \psi_n^{\circ -1}$ are critically finite branched coverings, they must be Thurston equivalent. Since $\psi_n \circ s_q \circ \psi_n^{\circ -1}$ and s_q are Thurston equivalent (by definition 2.1.1) it follows that f and s_q are Thurston equivalent. ■

The first example of a Julia set and corresponding lamination that will be looked at is Douady's rabbit polynomial, $f(z) \approx z^2 - 0.1226 + 0.7449i$, as illustrated in figure 2.5.

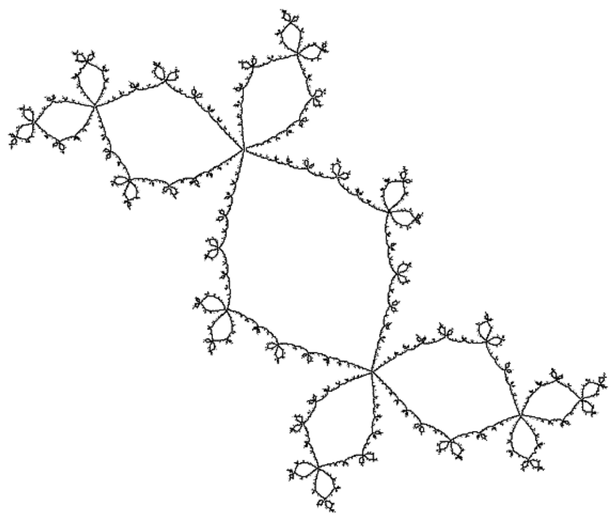

Figure 2.5: The Julia set for Douady's rabbit polynomial

Consider a finite sided gap in a lamination such as the highlighted triangle in figure 2.6. If the leaves bounding this gap have m points on S^1 as vertices then the gap, union its boundary leaves, separates the interior of the disc into m connected components. As all vertices of this gap map to the same point on the corresponding Julia set this gap is mapped to a point on the Julia set with m adjacent bounded Fatou components. Hence, the highlighted triangle in the lamination in figure 2.6 is mapped to the point adjacent to the three bounded Fatou components containing the orbit of the critical point.

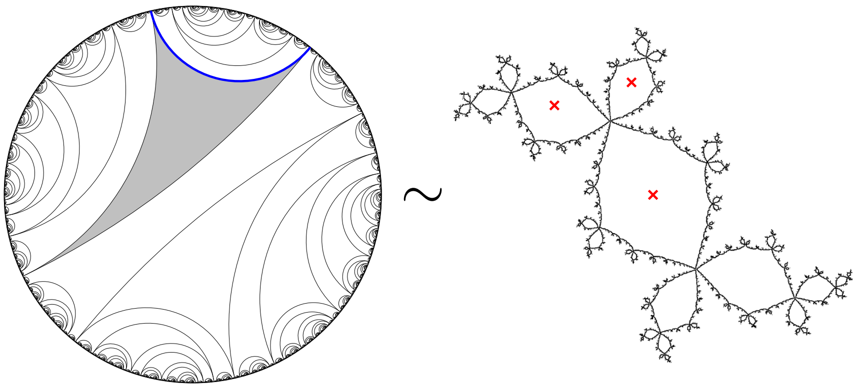

Figure 2.6: The Julia set of the rabbit and the corresponding lamination with minor leaf $\mu_{1/7}$ joining 1/7 to 2/7. An approximation of the post critical set is marked with crosses in the Fatou components.

Another example is the 'basilica'; the Julia set of $f(z) = z^2 - 1$. The corresponding lamination is shown in figure 2.7. In this case the highlighted minor leaf is not the side of a finite sided gap and separates the interior of the unit disc into two connected components. This minor leaf is mapped to the point on the Julia set adjacent to the two bounded Fatou components containing the post critical set of the basilica map.

Given a map f which is the centre of a hyperbolic component, and its corresponding Julia set, the minor leaf of the appropriate lamination has end-points with rotation equal to that of the angles of the two parameter rays landing at the root of the hyperbolic component containing f.

The *quadratic minor lamination*, or *QML*, is defined by Thurston to be the union of S^1 with all minor leaves. While it is not a lamination according to the definition given above (there is no associated map) it is interesting

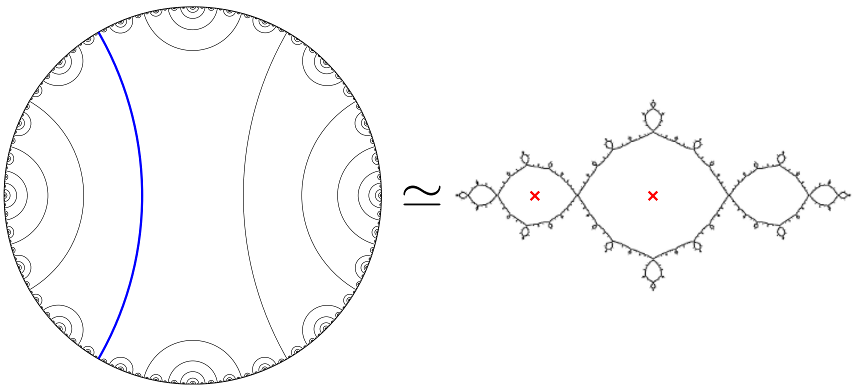

Figure 2.7: The Julia set of the basilica and the corresponding lamination with minor leaf $\mu_{1/3}$ which connects 1/3 to 2/3. An approximation of the post critical set is marked in the Fatou components.

for other reasons. The QML is conjecturally equivalent to the Mandelbrot set; this conjecture being equivalent to that stating that the Mandelbrot set is locally connected. Lavaurs outlines a simple algorithm for generating the QML in [LA]. A much faster algorithm is given in [TH]; an implementation of which accompanies this thesis, see appendix A.

It is possible to impose a partial ordering on the minor leaves present in the QML. Given two leaves, ℓ_a and ℓ_b, we say that

$$\ell_b > \ell_a$$

if, and only if, ℓ_a separates ℓ_b from 0. That is, ℓ_a separated the unit disc into two connected components, one of which contains 0 while the other contains ℓ_b.

In [TH] the following result is given regarding ordered leaves in the QML.

Lemma 2.2.5 *Suppose ℓ_m and ℓ_n are two non-degenerate leaves, with an endpoint at m and n respectively, in the QML with $\ell_m > \ell_n$ and ℓ_m periodic. Then ℓ_n is present in L_m.*

The concept of tuning, introduced in definition 2.0.11, is easily applied to laminations. If we have two lamination maps, s_f and s_g, then we may define the tuning $s_f \vdash s_g$ simply once we have found neighbourhoods of v_f, U_1 and U_2, and neighbourhoods of the orbit of v_g, V_1, and V_2, such that $\overline{U_1} \subset U_2$,

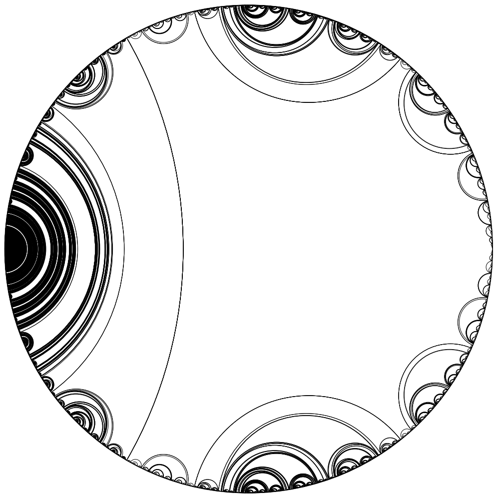

Figure 2.8: An approximation of the Quadratic Minor Lamination containing only periodic minor leaves

$s_f^{\circ n_f}(U_1) = U_2$, $\overline{V_1} \subset V_2$, and $s_g(V_1) = V_2$. Set $V_1 = \{z : |z| < 1 + \varepsilon_1\}$ for some small $\varepsilon_1 > 0$. Then, take $\varepsilon_2 = 2\varepsilon_1 + \varepsilon_1^2$ so that for $V_2 = \{z : |z| < 1 + \varepsilon_2\}$ we have that $s_g(V_1) = V_2$. Similarly, for g_f the infinite sided gap in L_f containing v_f, define $U_1 = \{z : \mathrm{dist}(z, g_f) < \varepsilon_3\}$ for some $\varepsilon_3 > 0$ and $U_2 = s_f^{\circ n_f}(U_1)$. The map s_f may be defined so that $\overline{U_1} \subset U_2$ meaning that the tuning can be constructed as in the statement of definition 2.0.11.

As the ε_is can be taken to be arbitrarily small, the lamination L_r, where $s_r = s_f \vdash s_g$, in the limit, is equal to a copy of L_f with g_f replaced with a copy of L_g. There exists a semi-conjugacy, φ, conjugating the action of $s_f^{\circ n_f}$ on $\overline{g_f}$ to that of $z \mapsto z^2$ on $\{z : |z| \leq 1\}$. By considering the pre-period of leaves on the boundary of g_f it is clear that the the periodic leaf μ_f is sent to the periodic point 0 by φ, the non-periodic pre-image of μ_f, of pre-period one, is sent to the point 1/2, of pre-period one, and so on. In this way not only do we see how L_g is embeded into g_f but also that the leaves to be added do not intersect any leaves of L_f – as the leaves of the boundary are sent to pre-image of 0 by φ. Hence, $L_f \subsetneq L_r$.

Lemma 2.2.6 *If s_p is not a tuning of any s_r ($r \neq 0$) and ℓ is any periodic* *leaf in* L_p *then the subset of* L_p

$$B = \bigcup_{i \geq 0} s^{\circ -i}(\ell)$$

is dense in L_p.

Proof: It must be that $\overline{B}$ is a clean invariant lamination contained in L_p. If it is not L_p then it must be L_r for some odd denominator rational $r \neq p$ because it has an infinite sided central gap. Then, as no leaf in L_p intersects a leaf in the forward orbit of ℓ, s_p is a tuning of s_r, a contradiction. ■

2.3 Matings

Previously we have extended polynomials onto the Riemann sphere, $\overline{\mathbb{C}}$, to apply the theory to put the Julia sets of the polynomials into the form of laminations. Because the laminations are on the sphere we may just as easily think of them on the *outside* of the unit disc as on the inside, by using the coordinate $w = \frac{1}{z}$. One difference when considering a lamination on the outside of the disc is the labels of the endpoints now label the rotation with respect to the opposite, clockwise, orientation. For example, the point previously labelled 3/4 would be labelled 1/4 on a lamination on the exterior of the disc. Labelling points using the anti-clockwise rotation is referred to as 'inside labelling' whereas using the clockwise labelling is 'outside labelling'. Figure 2.9 shows an example of a lamination on the outside of the disc; the left lamination uses the usual inside labelling while the right lamination uses the outside labelling, as might be expected.

The notion of *mating* was first discussed in [DH2]. Definition 2.3.1 is more suited to this work and is presented in [R2].

Definition 2.3.1 *Given two odd denominator rationals* r *and* t *in* $(0,1)$*, the* mating *of* s_r *and* s_t*, denoted* $s_r \perp\!\!\!\perp s_t$*, is defined to be*

$$s_r \perp\!\!\!\perp s_t = \begin{cases} s_r(z) & : |z| \leq 1, \\ s_t(z^{-1})^{-1} & : |z| \geq 1. \end{cases}$$

The mating $s_r \perp\!\!\!\perp s_t$ *has* mated lamination, $L_r \cup L_t^{-1}$.

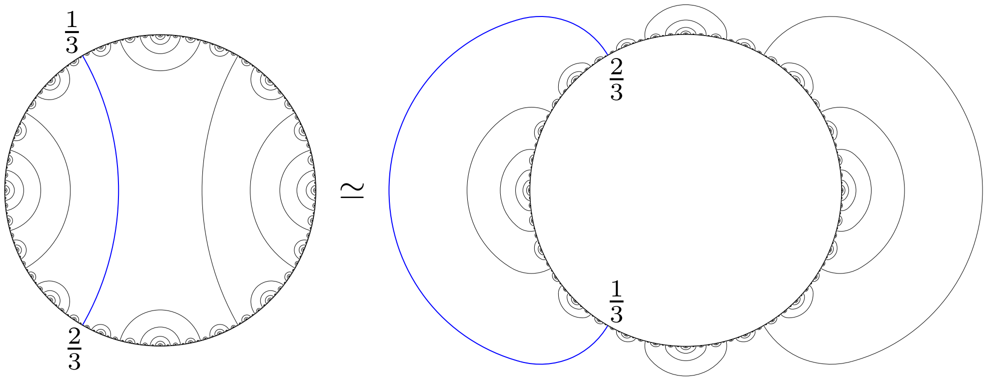

Figure 2.9: The basilica lamination on the interior and exterior of the disc

Definition 2.3.2 *A mating is said to be* admissible *if there exists a rational map R which is Thurston equivalent to F.*

The popularity of matings as a class of branched coverings (up to Thurston equivalnce) stems from the fact that many matings are admissible. From [TA] we have theorem 2.3.1, which uses work by Rees, building on Thurston's criterion (see [DH3]) to prescribe exactly when two maps can be mated to form an admissible mating. Here we state the theorem in the language of laminations

Theorem 2.3.1 (Tan Lei) *For odd denominator rationals r and p, the mating $s_r \perp\!\!\!\perp s_p$ is Thurston equivalent to a rational map if and only if there is no minimal minor leaf μ_t such that $\mu_t \leq \mu_p$ and $\mu_{1-t} \leq \mu_r$, that is, μ_p and μ_r are not in conjugate combinatorial limbs.*

The following is given in [R2].

Proposition 2.3.2 *If r and p are odd denominator rationals and φ_∞ is the semiconjugacy of proposition 2.1.3 between $s_r \perp\!\!\!\perp s_p$ and f, for a critically periodic rational map f, then $\varphi_\infty^{-1}(x)$ is a single equivalence class for the equivalence relation generated by $L_r \cup L_p^{-1}$, that is, the smallest equivalence relation such that x and y are equivalent if they are in the closure of the same leaf, or the closure of the same finite-sided gap of $L_p \cup L_r^{-1}$.*

2.3.1 Equivalence of Matings

To explore how matings relate to the parameter space of rational maps we need to understand which matings are equivalent. To approach this task a second invariant circle (sometimes referred to as a second equator) is employed.

Definition 2.3.3 *An* invariant circle *on a critically finite degree two branched covering, f, is a simple closed loop which separates the two critical orbits, and is such that, for $X(f)$ the post-critical set of f, $f^{-1}(\gamma)$ is connected and isotopic to γ in $\overline{\mathbb{C}} \setminus X(f)$, and also has the property that $f : f^{-1}(\gamma) \to \gamma$ preserves orientation.*

Lemma 2.3.3 *Suppose that f is a critically periodic degree two branched covering with post-critical set $X(f)$. Suppose that $\gamma \subset \overline{\mathbb{C}} \setminus X(f)$ is an invariant circle on f. Let D_1 and D_2 be the two components of $\overline{\mathbb{C}} \setminus f^{-1}(\gamma)$ then f is Thurston equivalent to $s_r \amalg s_p$, where s_r and s_p are equivalent to g and h respectively, where $g = f$ on $D_1 \cup f^{-1}(\gamma)$ and g has a single fixed critical point in D_2, and h is similarly defined, with the roles of D_1 and D_2 reversed.*

Proof: Up to Thurston equivalence, we can assume that γ is the unit circle and that this is preserved by f. We can also assume that $f(z) = z^2$ on the unit circle, and that $g(z) = z^2$ outside the unit circle, and $h(z) = z^2$ inside the unit circle.

Using theorem 2.1.5 and lemma 2.2.4 we see that there exist lamination maps s_r and s_p which are Thurston equivalent to to g and h respectively. Using Definition 2.1.1 of Thurston equivalence, we can assume that all the branched coverings in the homotopy g_t between g and s_r (or the homotopy h_t between h and $z \mapsto (s_p(z^{-1}))^{-1}$) are given by $z \mapsto z^2$ on the unit circle. Then we have an equivalence f_t between f and $s_r \amalg s_p$ where

$$f_t(z) = \begin{cases} g_t(z) & : |z| \leq 1, \\ h_t(z) & : |z| \geq 1 \end{cases}$$

■

Given a mating between two polynomials there may exist an equivalent mating. This happens when the mated lamination resulting from the mating of the two polynomial laminations has a second invariant circle which contains one of the critical orbits.

For example, the mating

$$s_{3/7} \perp\!\!\!\perp s_{1/7}$$

between the aeroplane and the rabbit polynomials has the lamination shown in figure 2.10. In this mating the orbit of $\mu_{1/7}$ on the exterior of S^1 forms a triangle, with endpoints at 1/7, 2/7 and 4/7 in the outside labelling, which is fixed by the mating map (this is the lower of the two large triangles in the figure). The gaps of $L_{3/7}$, on the interior of S^1, which share these vertices contain the critical orbit of $L_{3/7}$.

Figure 2.11 highlights a second invariant circle present in $L_{3/7} \cup L_{1/7}^{-1}$ (up to homotopy with respect to the critical orbits). This closed loop is homotopic to a circle and, under homotopy preserving the critical points of the mating, is backward and forward invariant with respect to the lamination map. See chapter 3 for a more detailed description of this example.

Lemma 2.3.3 gives that the existence of a second invariant circle gives rise to another mating, in the case of the example presented in figure 2.11, $s_{1/7} \perp\!\!\!\perp s_{3/7}$, which is Thurston equivalent to the first. Matings which are equivalent to matings consisting of the same constituent polynomials, but in the reverse order, are known as Wittner flip matings, with reference to [W]. If a mating has an equivalent mating, we say it is a *shared* mating.

For much of this thesis a particular example of matings will be considered. Due to the construction of the algorithm discussed in chapter 3 only matings of the form $s_{3/7} \perp\!\!\!\perp s_p$, where $\mu_p > \mu_{1/7}$, are discussed throughout the rest of this document. The following theorem will be useful.

Lemma 2.3.4 *Suppose that f is a critically periodic degree two branched covering with post-critical set $X(f)$ which is equivalent to a rational map. Suppose there are two isotopically invariant circles γ_1 and γ_2 in $\overline{\mathbb{C}} \setminus X(f)$, that is, both satisfying the conditions of γ in 2.3.3. Let $s_{r_1} \perp\!\!\!\perp s_{p_1}$ and $s_{r_2} \perp\!\!\!\perp s_{p_2}$ be the matings associated to γ_1 and γ_2 which are equivalent to f and let φ_0 and φ_1 be homeomorphisms which are isotopic via an isotopy constant on $X(s_{r_1} \perp\!\!\!\perp s_{p_1})$ such that*

$$\varphi_0 \circ (s_{r_1} \perp\!\!\!\perp s_p) = (s_{r_2} \perp\!\!\!\perp s_q) \circ \varphi_1$$

$$\varphi_0(X(s_{r_1})) = X(s_{r_2}), \quad \varphi_0(X(s_p)^{-1}) = X(s_q)^{-1}.$$

Suppose also that γ_1 and γ_2 are isotopically distinct. Then either $s_{r_1} \neq s_{r_2}$ or $s_p \neq s_q$.

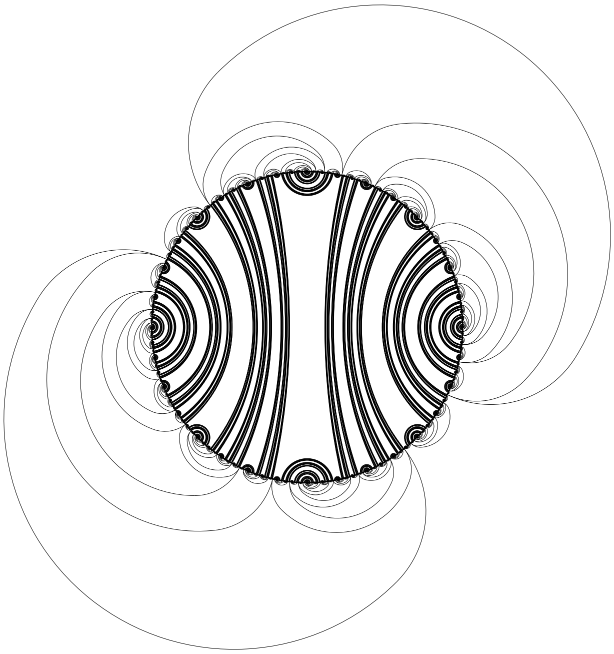

Figure 2.10: $L_{3/7} \perp\!\!\!\perp L_{1/7}$.

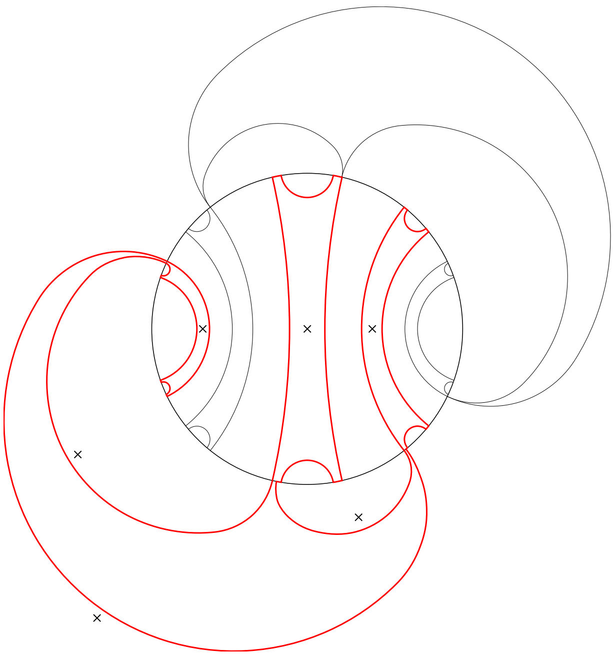

Figure 2.11: A second invariant circle (red) drawn up to homotopy with respect to the marked critical orbit.

Proof: This follows from the uniqueness statement of Thurston's Theorem, which, if properly interpreted, says that there is no non-trivial self-equivalence of a critically finite branched covering which is equivalent to a rational map. In terms of Definition 2.1.1 this means that if $(f_0, X_0) \simeq (f_1, X_1)$ then φ_0 and φ_1 must be isotopic to the identity relative to X_0 and $f_0^{-1}(X_0)$ respectively. ■

Lemma 2.3.5 *Consider the mating $s_{3/7} \amalg s_p$, with $\mu_p > \mu_{1/7}$. There exists a lamination map s_q such that*

$$s_{3/7} \amalg s_p \simeq s_{1/7} \amalg s_q.$$

Proof: As $\mu_p > \mu_{1/7}$ lemma 2.2.5 gives that all the structure depicted in figure 2.11 is present in $L_{3/7} \cup L_p^{-1}$. Let $\gamma_1 = S^1$ and γ_2 be the second invariant circle highlighted in figure 2.11 (up to homotopy with respect the the post-critical set of $s_{3/7} \amalg s_p$). Both γ_1 and γ_2 have the properties of γ in lemma 2.3.3 so that lemma 2.3.4 gives the existence of the mating $s_r \amalg s_q$ which is equivalent to $s_{3/7} \amalg s_p$ and

$$\varphi_0 \circ (s_{3/7} \amalg s_p) = (s_r \amalg s_q) \circ \varphi_1,$$

$$\varphi_0(X(s_{3/7})) = X(s_r), \quad \varphi_0(X(s_p)^{-1}) = X(s_q)^{-1}.$$

We may also assume that $\varphi_0(\gamma_2) = \gamma_1$.

To obtain $r = \frac{1}{7}$ construct a set of three arcs ζ_i, $1 \leq i \leq 3$, with endpoints on γ_2 close to the vertices of the red triangle in figure 2.11. This set of arcs is clearly isotopic to a set of pre-images under $s_{3/7} \amalg s_p$. It follows that we can choose the homeomorphism φ_0 to map the arcs ζ_i close to the sides of the triangle with vertices at $e^{2\pi i(1/7)}$, $e^{2\pi i(2/7)}$, $e^{2\pi i(4/7)}$, and map $X(s_{3/7})$ to $X(s_{1/7})$. It follows that $\varphi_0 \circ (s_{3/7} \amalg s_p)$ is isotopic to $s_{1/7}$ on the unit disc, via an isotopy which is constant on $X(s_{1/7})$. It follows from definition 2.1.1 that $s_{3/7} \amalg s_p$ is Thurston equivalent to $s_{1/7} \amalg s_q$ for some q. ■

Definition 2.3.4 *Let X_p be the union of the postcritical set of $s_{3/7} \amalg s_p$ and a fixed point in the fixed triangle of L_p^{-1}. Let X_q be the union of the postcritical set of $s_{1/7} \amalg s_q$ and a fixed point in the fixed triangle of $L_{1/7}$. Then we may choose φ_0 such that*

$$(s_{3/7} \amalg s_p, X_p) \simeq_{\varphi_0} (s_{1/7} \amalg s_q, X_q).$$

For future reference we call these fixed points β^*_{-1} *and* γ^*_{-1} *respectively.*

Throughout the rest of this document these second invariant circles are exploited to calculate equivalent matings. Symbolic dynamics is used to partition the mated laminations and label the leaves and gaps of the second invariant circle. The details of the algorithm which has been developed are discussed in chapter 3.

Chapter 3

An algorithm for finding equivalent matings

In this chapter we describe an algorithm which finds a mating equivalent to the given mating where the equivalence arises from the Wittner flip described in section 2.2.1. Also, we show that the algorithm does converge but do not attempt to give any estimate on the time necessary. The algorithm requires, as input, a mating of the form $s_{3/7} \perp\!\!\!\perp s_p$, where $\mu_p > \ell_{1/7,2/7}$. That is, a mating between the aeroplane on the inside of S^1 and a polynomial in the rabbit limb of the Mandelbrot set on the outside. As a consequence of the mechanism involved, a mating of the form $s_{1/7} \perp\!\!\!\perp s_q$ must be output, with q described in more detail in chapter 4.

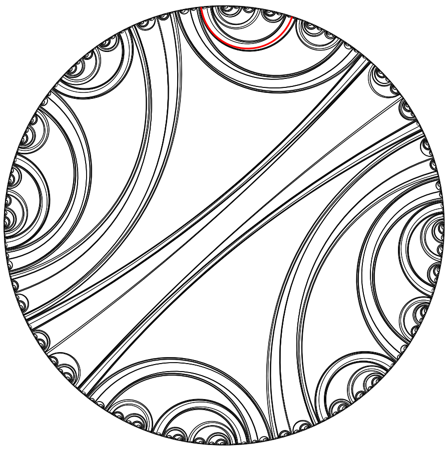

Figure 3.1: $L_{1/5}$ $(\mu_{1/5} > \mu_{1/7})$

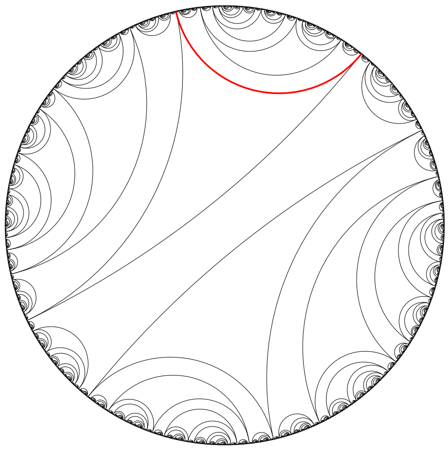

Figure 3.2: $L_{1/7}$

Throughout this section, and the rest of this document, the periodic triangle, with vertices 1/7, 2/7 and 4/7, of $L_{1/7}$, the rabbit lamination, will be referred to as T. Its immediate pre-image will similarly be labelled $-T$. Both T and $-T$ are present in any L_p, with $p \in (1/7, 2/7)$, as a consequence of lemma 2.2.5.

Next it is shown how to construct the second invariant circle γ_2, of lemma 2.3.5 (and illustrated in figure 2.11). Using the triangles in L_p which are pre-images of T we may construct a second invariant circle on the mated lamination $L_{3/7} \cup L_p^{-1}$. Let C_0' be the union of T and the three gaps of $L_{3/7}$ containing the critical orbit. Let

$$C_k' = (s_{3/7} \amalg s_p)^{\circ -k}(C_0').$$

For all finite k there is an $\varepsilon > 0$ such that if we define

$$C_k'' = \{z \in \overline{\mathbb{C}} : d(z, C_k') < \varepsilon\}$$

then the boundary, C_k, of C_k'' is a simple loop and a second invariant circle. The loop γ_2 of lemma 2.3.5 is equal to C_0. The second invariant circle C_k is defined up to homotopy with respect to the critical orbits; varying k gives different invariant circles which are all equivalent up to this homotopy. Hence, we may simply discuss the second invariant circle C, without reference to k.

Let φ_0 and φ_1 be defined as in lemma 2.3.5 so that

$$\varphi_0 \circ (s_{3/7} \amalg s_p) = (s_{1/7} \amalg s_q) \circ \varphi_1,$$

$$\varphi_0(X(s_{3/7}) = X(s_{1/7}), \quad \varphi_0(X(s_p)^{-1}) = X(s_q)^{-1},$$

and

$$\varphi_0(C_0) = S^1.$$

Letting φ_n be defined as in lemma 2.1.2 we have that

$$\varphi_n(C_n) = S^1.$$

Let D_n be the disc bounded by C_n. We see in the proof of lemma 2.3.5 that φ_0 has been defined to map a neighbourhood of T in D_0 to a neighbourhood of T in the unit disc. This gives that φ_n maps a neighbourhood of $(s_{3/7} \amalg s_p)^{\circ -n}(T)$ in D_n to a neighbourhood of $(s_{1/7} \amalg s_q)^{\circ -n}(T)$ in the unit disc.

Given this correspondence between triangles of the two models the rest of this chapter focusses on finding the value of q.

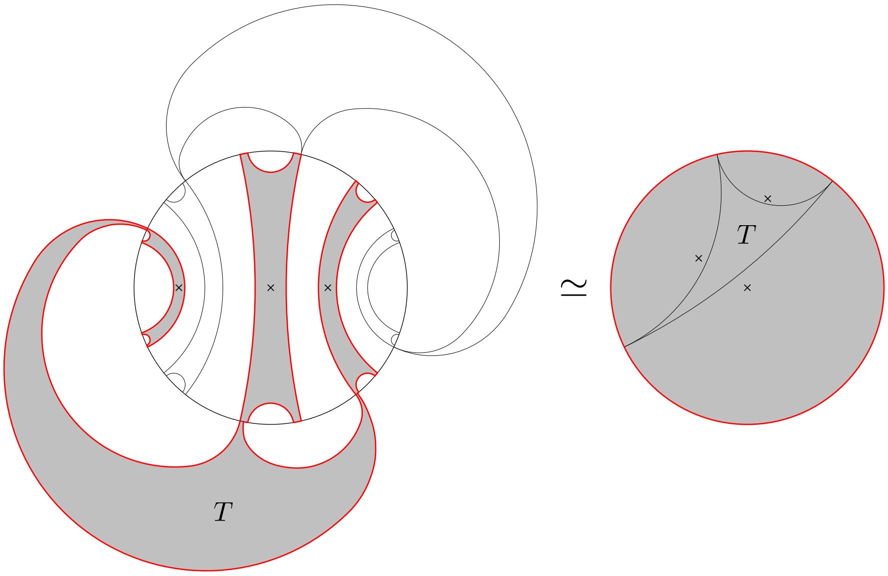

Figure 3.3: How to identify the triangle T across the equivalence $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$. Here the second circle is C_0.

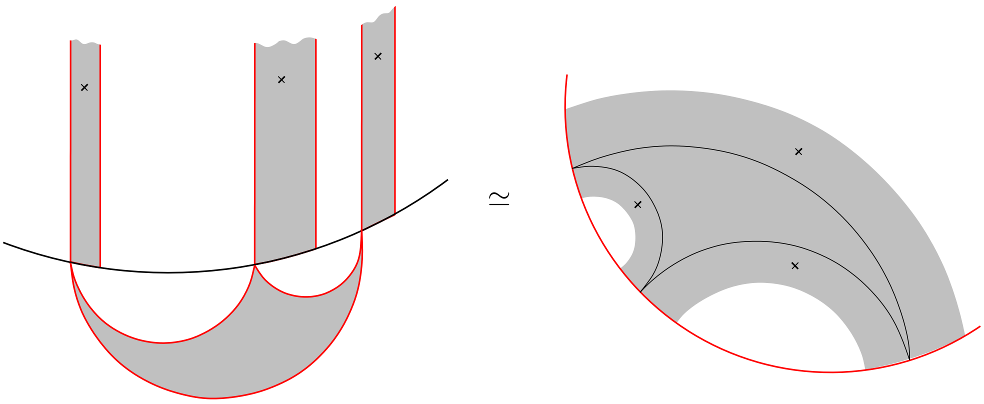

Figure 3.4: Correspondence between triangles in $L_{3/7} \cup L_p^{-1}$ (left) and $L_{1/7} \cup L_q^{-1}$ (right). Here the marked points are pre-images of the critical orbits of $L_{3/7}$ (left) and $L_{1/7}$ (right).

3.0.2 Convergence of the Algorithm

In this short section we aim to prove theorem 3.0.6 which underpins the algorithm discussed in section 3.1. Before stating theorem 3.0.6 we require a number of definitions.

Recall definition 2.3.4 which defines β^*_{-1} to be a point in T, on the exterior of S^1, which is fixed by $s_{3/7} \perp\!\!\!\perp s_p$. The sets X_p and X_q are also introduced in the same definition.

Definition 3.0.5 *(illustrated in figure 3.5)*

- *Take z_{-1} to be a point in the orbit of ∞ (the second critical orbit of $s_{3/7} \perp\!\!\!\perp s_p$) such that infinite sided gap containing a point in the critical orbit separates z_{-1} from T.*
- *Choose β_{-1} to be a straight path connecting β^*_{-1} and z_{-1}.*
- *Take ℓ_{-1} to be the periodic leaf on the boundary of the infinite sided gap containing z_{-1}.*
- *Define $\Delta_{-1} = T$.*

Inductively, define z_n, ℓ_n, β^*_n, β_n and Δ_n as follows:

- *Take z_{n+1} to be the periodic pre-image of z_n.*
- *Take ℓ_{n+1} be the periodic pre-image of ℓ_n.*
- *Let Δ_{n+1} be the pre-image of Δ_n closest to ℓ_{n+1}.*
- *Let β^*_{n+1} be the pre-image of β^*_n contained in Δ_{n+1}.*
- *Let β_{n+1} be the component of $(s_{3/7} \perp\!\!\!\perp s_p)^{\circ -1}(\beta_n)$ with one endpoint at z_{n+1} and the other at β^*_{n+1}*

Theorem 3.0.6 *Let p be of period n_p under $x \mapsto 2x$ mod 1. Let φ_m be such that*

$$(s_{3/7} \perp\!\!\!\perp s_p, (s_{3/7} \perp\!\!\!\perp s_p)^{\circ -m} X_p) \simeq_{\varphi_m} (s_{1/7} \perp\!\!\!\perp s_q, (s_{1/7} \perp\!\!\!\perp s_q)^{\circ -m}(X_q))$$

Then if z_k is the critical value, $\lim_{m \to \infty} \varphi_{mn_p+k+1}(\beta^*_{mn_p+k})$ exists and is an endpoint of μ_q.

This will be proved shortly. We shall use lemma 3.0.7. For this lemma, we make the assumption that $\beta^*_{in_p+k}$ is the centroid of the triangle Δ_{in_p+k} and that the β_i are straight lines (while preserving the property $(s_{3/7} \perp\!\!\!\perp s_p)^{\circ -1}(\beta_{in_p+k}) = \beta_{(i+1)n_p+k})$. Note that $z_{in_p+k} = z_k$ is fixed and $\ell_{in_p+k} = \ell_k = \mu_p$.

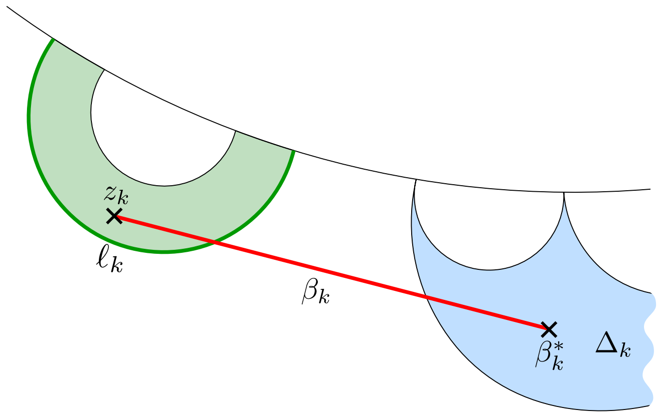

Figure 3.5: A visual guide to definition 3.0.5, with Δ_k in blue and β_k in red.

Lemma 3.0.7 *For z_k the pre-image of z_{-1} which is also a critical value of $s_{3/7} \perp\!\!\perp s_p$, the limit*

$$\lim_{i\to\infty} \beta_{in_p+k}$$

exists and consists of a path connecting a point on μ_p to z_k.

Proof: If g_k is the gap containing z_k we have that $\beta_{in_p+k} \setminus g_k$ pulls back homeomorphically, as it crosses no other gap in the orbit of g_k. Hence, the length of $\beta_{in_p+k} \setminus g_k$ tends to 0 under repeated applications of $(s_{3/7} \perp\!\!\perp s_p)^{-1}$.

The limit of Δ_{in_p+k}, as $i \to \infty$ is either a point or a leaf. Combining this with the fact that the length of $\beta_{in_p+k} \setminus g_k$ tends to 0 as $i \to \infty$ we see that the limit of Δ_{in_p+k} is either an endpoint of μ_p, or μ_p itself. For each of these possible limits result is immediate. ∎

Corollary 3.0.8 *The limit*

$$\lim_{i\to\infty} \beta^*_{in_p+k} \in \mu_p.$$

Proof: In the proof of lemma 3.0.7 it is observed that the limit of Δ_{in_p+k} is either an endpoint of μ_p, or μ_p itself. Under the assumption that β_n is the centroid of Δ_n the result holds. ∎

Proof: (of Theorem 3.0.6) Let f be the critically finite rational map which is Thurston equivalent to $s_{3/7} \perp\!\!\!\perp s_p$, and, hence, also to $s_{1/7} \perp\!\!\!\perp s_q$. Let φ'_m and φ''_m be the sequences of homeomorphisms, from lemma 2.1.2, giving Thurston equivalences between $s_{3/7} \perp\!\!\!\perp s_p$ and f, and $s_{1/7} \perp\!\!\!\perp s_q$ and f, respectively. Further, requiring

$$\varphi_0 = \varphi_0''^{-1} \circ \varphi_0',$$

gives also that

$$\varphi_n = \varphi_n''^{-1} \circ \varphi_n'.$$

Let φ'_∞ and φ''_∞ be the limiting semi-conjugacies given in proposition 2.1.3.

Lemma 3.0.7 gives that $\varphi'_\infty(\beta_{mn_p+k})$ converges to a path joining the critical value $\varphi'_\infty(z_k)$ to the unique point on the boundary of the attractive basin fixed by f^{n_p}, x_∞. Then by uniform convergence the paths $\varphi'_{mn_p+k+1}(\beta_{mn_p+k})$ also converge to x_∞. Also

$$\varphi_{m+1}(\beta^*_m) = \varphi_{m'}(\beta^*_m) \text{ for all } m' > m \tag{3.1}$$

and

$$\varphi'_{m+1}(\beta^*_m) = \varphi'_{m'}(\beta^*_m) \text{ for all } m' > m \tag{3.2}$$

and

$$\varphi''_{m+1}(\gamma^*_m) = \varphi''_{m'}(\gamma^*_m) \text{ for all } m' > m \tag{3.3}$$

where $\gamma^*_m = \varphi_{m+1}(\beta^*_m)$.

Considering the limit

$$\begin{aligned}
\lim_{m\to\infty} \varphi''_{mn_p+k+1}(\gamma^*_{mn_p+k}) &= \lim_{m\to\infty} \varphi'_{mn_p+k+1}(\beta^*_{mn_p+k}) \\
&= \varphi'_\infty(\lim_{n\to\infty} \beta^*_{mn_p+k}) \qquad \text{(by 3.2)} \\
&= \varphi'_\infty(\beta^*_\infty) \qquad \text{(for some } \beta^*_\infty \in \mu_p, \text{ by lemma 3.0.8)} \\
&= x_\infty.
\end{aligned}$$

Now, equation 3.3 shows that

$$\lim_{m\to\infty} \varphi''_{mn_p+k+1}(\gamma^*_{mn_p+k}) = \varphi''_\infty(\lim_{m\to\infty} \gamma^*_{mn_p+k}).$$

Hence, we have that

$$\lim_{m\to\infty} \gamma^*_{mn_p+k} \in \{z \in \overline{\mathbb{C}} : \varphi''_\infty(z) = x_\infty\} = \overline{\mu_q}$$

(as the points on $\overline{\mu_q}$ are the only points on the boundary of the gap containing the critical value of $s_{3/7} \perp\!\!\!\perp s_q$ on the exterior of S^1 (inclusive) which are fixed under $(s_{3/7} \perp\!\!\!\perp s_q)^{\circ n_p}$). ■

Definition 3.0.6 *Let δ_i be the triangle in $L_{1/7}$ which contains the point γ_i^*.*

Corollary 3.0.9 *The limit*

$$\lim_{i \to \infty} \delta_i$$

exists and is an endpoint of μ_q.

Proof: All triangles in $L_{1/7}$ pull back to points. Hence, this result follows immediately from theorem 3.0.6. ■

3.1 Executing the algorithm

A number of steps are involved in the algorithm. Here those steps are described in some detail. Throughout this section let $G_{3/7}$ be the periodic central gap of $L_{3/7}$ and $G_{1/7}$ the periodic central gap of $L_{1/7}$. To begin, we discuss a scheme for labelling the Δ_i.

In the previous section we have seen how to calculate the sequence of points, β_i^* which converge to a point on μ_p. We now focus on the sequence of triangles, $\{\Delta_i\}$, which contain these points.

Figure 3.6 illustrates a labelling of $L_{1/7}$ where

$$\begin{aligned}
L_1 &= (1/7, 2/7) \\
L_2 &= (2/7, 4/7) \\
UC &= (1/14, 1/7) \\
BC &= (4/7, 9/14) \\
R_1 &= (9/14, 11/14) \\
R_2 &= (11/14, 1/14) \\
L &= \overline{L_1 \cup L_2} \\
R &= \overline{R_1 \cup R_2}
\end{aligned}$$

We may label the triangles δ_i using the regions above. If a δ_i has all three vertices in one of the above regions we assign the label of that region to the δ_i. In this way, T is labelled by L, $-T$ by R and all other δ_i by one of L_1, L_2, UC, BC, R_1 or R_2.

Definition 3.1.1 *An* admissible word *on* $\{L_1, L_2, BC, UC, R_2, R_1\}$ *is any word*

$$W = w_n w_{n-1} w_{n-2} \dots w_2 w_1 w_0,$$

with $w_i \in \{L_1, L_2, BC, UC, R_2, R_1\}$, *where*

$$z \mapsto z^2 : w_i \to w_{i-1}, \quad \forall i.$$

As every δ_i has a label, we may form a word which encodes the forward itinerary of the δ_i under the mated lamination map $s_{1/7} \perp\!\!\!\perp s_q$. For example, if δ_i has label R_2 and δ_{i-1} has label BC then we see that the word storing the itinerary for δ_i begins

$$R_2 BC \dots$$

As every δ_i is an i^{th} pre-image of $T = \delta_{-1}$ we may assign a length $(i + 2)$ admissable word, on the characters

$$\{L_1, L_2, BC, UC, R_2, R_1, L, R\}$$

to each δ_i. Note that this word is enough to uniquely identify δ_i in $L_{1/7}$.

With reference to lemma 2.1.2 and definition 2.3.4 let φ_n be the homeomorphism giving

$$(s_{3/7} \perp\!\!\!\perp s_p, X_p^n) \simeq_{\varphi_n} (s_{1/7} \perp\!\!\!\perp s_q, X_q^n)$$

where X_p^n and X_q^n contain the post-critical set and β_{-1}^* to β_n^* and γ_{-1}^* to γ_n^* respectively. Lemma 2.3.5 gives that φ_n exists. In the first section of this chapter we have seen that φ_n maps a neighbourhood of Δ_i onto a neighbourhood of a unique δ_i. Using this correspondence between triangles under φ_n we may consider the labelling to occur on $\varphi_n^{-1} \circ s_{1/7} \circ \varphi_n$ and label the Δ_is directly. Hence, each Δ_i can be assigned a length $(i + 1)$ admissable word of the labelling $\{L_1, L_2, BC, UC, R_2, R_1, L, R\}$ of $L_{1/7}$.

Figure 3.7 shows a sample of the correspondence between the Δ_i and δ_i.

Finding which label should be attached to each of the triangles Δ_i is the most involved step in using this algorithm. It is noted on page 43 that if

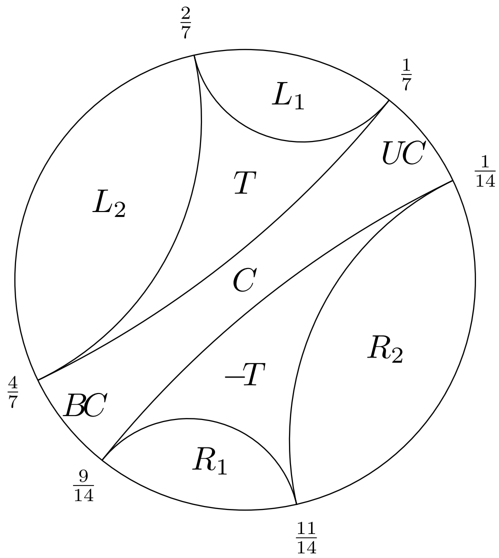

Figure 3.6: Labelling of $L_{1/7}$, the rabbit.

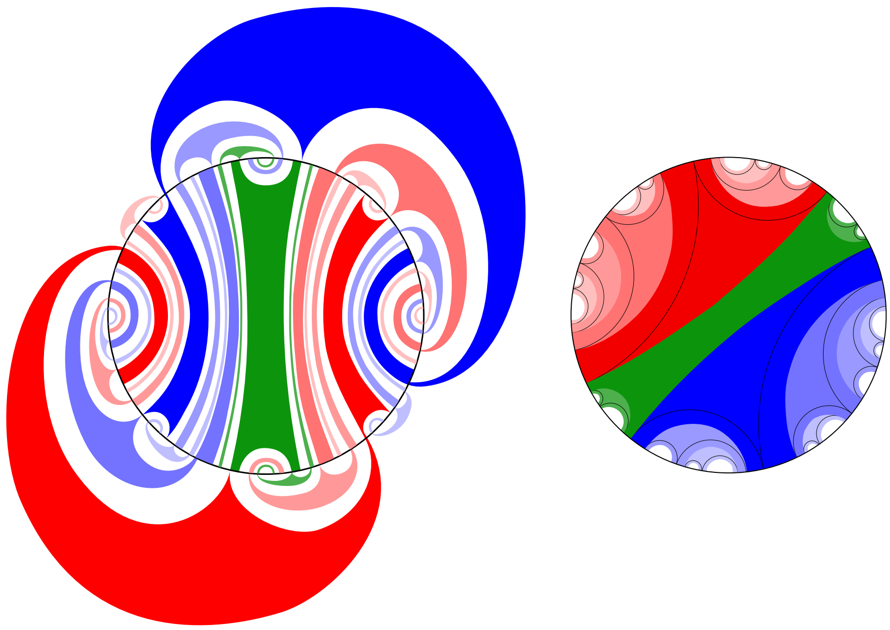

Figure 3.7: $L_{3/7} \cup L_p^{-1}$ on the left with regions on the interior of the second invariant circle highlighted. On the right is the image of the second invariant circle under the Thurston equivalence with the highlighting preserved. Label red triangles L, label blue triangles R, label green triangles UC or BC

we can label Δ_i with L, R, UC or BC in the labelling of $L_{1/7}$ then this is enough to determine the full labelling discussed above. We can do this by noting whether a path contained inside the second invariant circle connecting Δ_i to $G_{3/7}$ necessarily passes through T (then label Δ_i as L), $-T$ (then label Δ_i as R), or UC or BC (then label Δ_i as UC or BC, respectively). To do this it is possible to simply calculate $L_{3/7} \cup L_q$ to the required level of detail, labelling triangles as they are calculated. However, as the period of the minor leaves increases this method quickly becomes intractable. To circumvent this problem a tableau is employed.

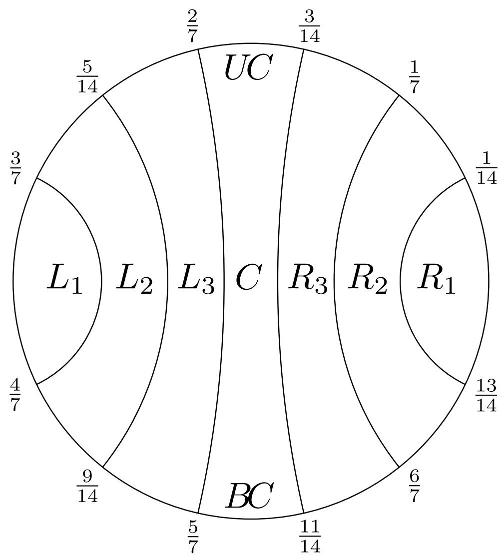

Figure 3.8: Labelling of $L_{3/7}$, the aeroplane.

The tableau utilises symbolic dynamics on the symbols L_1, L_2, L_3, C, UC, BC, R_3, R_2, and R_1 on $L_{3/7}$. These labels represent regions on S^1 as illustrated in figure 3.8 and are acted on by $s_{3/7}$. A word constructed of these labels

$$X = x_n x_{n-1} x_{n-2} \ldots x_1 x_0$$

represents all points

$$\{z \in S^1 : s_p^i(z) \in x_{n-i},\ 0 \leq i \leq n\}.$$

Finite length words represent a union of intervals on S^1 whereas infinite length words represent a union of points on S^1. Define a second word

$$Y_1 = y_m y_{m-1} \dots y_1 X.$$

Then Y_1 represents an m^{th} pre-image of X. Similarly, if

$$Y_2 = X y_{m-1} y_{m-2} \dots y_0$$

then Y_2 represents a subset of X.

The region denoted by C is bounded by the endpoints of the major leaves of $L_{3/7}$. Hence, any word in the labelling of $L_{3/7}$ ending in C specifies pre-images of these endpoints which, in turn, specify pre-images of the major leaves. As leaves in $L_{3/7}$ may only lie in the boundary of one infinite sided gap these pre-images of the major leaves specify a unique pre-image of $G_{3/7}$.

Words ending in either L_2, R_1 or C may be associated with a pre-image of $G_{3/7}$. For a word W ending in L_2 define $W' = WC$. If W ends R_1 then define $W' = WR_2C$. If W ends in C then let $W' = W$. Then W' specifies a unique pre-image of $G_{3/7}$, as described above. We associate this gap with W.

3.1.1 The Tableau

The tableau is an array consisting of entries

$$x_{i,j}, \quad i \geq 0, \quad j \geq r_i$$

where the $x_{0,j}$ form the top row and the $x_{i,0}$ the right-most column. Further,

$$\begin{aligned}
x_{i,j} &\in \{L_1, L_2, L_3, C, UC, BC, R_3, R_2, R_1\} \quad (\text{of } L_{3/7}),\\
x_{i,j-1} &\subset s_{3/7}(x_{i,j}),\\
r_0 &= 0,\\
r_i &< r_{i+1} \text{ for } i > 0,\\
x_{i,r_i} &= C \text{ for } i > 0.
\end{aligned}$$

To generate the tableau we first compute the complete top row. To generate this row we take ν_j to be the vertex of Δ_j which is closest to ℓ_j (in the euclidean metric on S^1) with

$$s_{3/7} \perp\!\!\!\perp s_p(\nu_{j+1}) = \nu_j.$$

Set $x_{0,j}$ to be the label of the region containing ν_j.

Note: As we start with a sequence of triangles in $L_{3/7} \cup L_p^{-1}$ which converge to μ_p *any* vertex will converge to an endpoint of μ_p. Hence, any vertex is sufficient to generate the tableau; the choice of vertex for ν_j seen here is merely a convenient suggestion which is assumed in this discussion.

We define r_i inductively. Given a complete i^{th} row define $r_{i+1} > r_i$ such that

$$\begin{aligned} x_{i,j} &\in \{L_1, C, UC, BC, R_2\} \text{ for } r_i \leq j < r_{i+1} \\ x_{i,r_{i+1}} &\in \{L_2, R_1\}. \end{aligned}$$

We assign the label $c_j \in \{UC, BC, L, R\}$ to the j^{th} column of the tableau as follows. For $i(j)$ such that $r_{i(j)} < j \leq r_{i(j)+1}$ let

$$c_j = \begin{cases} L & \text{if } x_{i(j),j} \in \{L_2, R_1\} \\ R & \text{if } x_{i(j),j} \in \{L_1, R_2\} \\ UC & \text{if } x_{i(j),j} = UC \\ BC & \text{if } x_{i(j),j} = BC. \end{cases}$$

Let the word $x_{i,j}x_{i,j-1}\ldots x_{i,r_i}$ be associated with the gap $G_{i,j}$, an infinite sided pre-image of $G_{3/7}$. We have seen that

$$x_{i,j} \in \{L_1, C, UC, BC, R_2\} \text{ for } r_i \leq j < r_{i+1},$$

which gives that $G_{i,r_{i+1}}$ is equal to one of the three periodic gaps of $L_{3/7}$ (recall section 2.3.1 and figures 2.10 and 2.11). Hence, the gap $G_{i,r_{i+1}}$ shares a vertex with $-T$. Then the triangle $-T$ joins $G_{i,r_{i+1}}$, in $x_{i,r_{i+1}}$, to $G_{3/7}$, in $x_{i+1,r_{i+1}} = C$. Set

$$\begin{aligned} \Delta_{i,r_{i+1}} &= -T \\ \nu_{i,r_{i+1}} &= \partial G_{i,r_{i+1}} \cap \partial(-T) \\ \nu'_{i,r_{i+1}} &= 3/14 \ (= \partial G_{i+1,r_{i+1}} \cap \partial(-T)) \end{aligned}$$

Then define $\nu_{i,j+1}$, $\Delta_{i,j+1}$ and $\nu'_{i,j+1}$ recursively for $j \geq r_{i+1}$ such that

$$\nu_{i,j+1} \in x_{i,j+1}, \quad s_{3/7} \amalg s_p(\nu_{i,j+1}) = \nu_{i,j},$$

$$s_{3/7} \amalg s_p(\Delta_{i,j+1}) = \Delta_{i,j}, \quad \nu_{i,j+1} \subset \partial\Delta_{i,j+1},$$

$$\nu'_{i,j+1} \in \partial\Delta_{i,j+1} \cap S^1, \quad s_{3/7} \amalg s_p(\nu'_{i,j+1}) = \nu'_{i,j}.$$

Setting $x_{i+1,j}$ to be the label of the region containing $\nu'_{i+1,j}$ completes the $(i+1)^{\text{th}}$ row of the tableau.

The above shows that $G_{i,j}$ is connected to $G_{i+1,j}$ by $\Delta_{i,j}$, a pre-image of $-T$. Hence, a column consists of infinite sided gaps connected by leaves of $L_{1/7}$. When we set c_j we are labelling the infinite sided gap $G_{i(j),j}$, for $i(j)$ such that $r_{i(j)} < j \leq r_{i(j)+1}$, with respect to the labelling of $L_{1/7}$ on the second invariant circle. However, as $G_{i(j),j}$ and $G_{0,j}$ are connected by a chain of leaves in $L_{1/7}$ and infinite sided gaps, not equal to $G_{3/7}$, in $L_{3/7}$ we have that $G_{0,j}$ must also lie in the region labelled by c_j in the labelling of $L_{1/7}$ on the second invariant circle.

It is immediate from the structure of $L_{1/7}$ (shown in figure 3.6) that the angle doubling map sends the labelled regions of $L_{1/7}$ to each other by

$$\begin{aligned} UC, BC &\rightarrow L_1 \\ L_1, R_1 &\rightarrow L_2 \\ L_2, R_2 &\rightarrow BC \cup R_1 \cup R_2 \cup UC. \end{aligned}$$

This gives a way to translate from the $c_j \in \{L, R, UC, BC\}$ into $c'_j \in \{L_1, L_2, R_1, R_2, UC, BC\}$. Letting

$$\begin{aligned} c_j = L &\Rightarrow c'_j = \begin{cases} L_1 & \text{if } c_{j-1} = L, \\ L_2 & \text{otherwise,} \end{cases} \\ c_j = R &\Rightarrow c'_j = \begin{cases} R_1 & \text{if } c_{j-1} = L, \\ R_2 & \text{otherwise,} \end{cases} \\ c_j = UC &\Rightarrow c'_j = UC, \\ c_j = BC &\Rightarrow c'_j = BC, \end{aligned}$$

gives that $\Gamma_k = c'_k c'_{k-1} \dots c'_0$ is the word which labels $G_{0,k}$ with respect to the labelling of $L_{1/7}$ on the second invariant circle.

As $G_{0,k}$ shares a vertex with Δ_k there must be a gap $g_k \subset L_{1/7}$ which shares a boundary leaf with δ_k with all vertices of g_k in the region labelled by Γ_k. As

$$g_i \cap S^1 \in \Gamma_i,$$

setting $\Gamma_\infty = \lim_{i\to\infty} \Gamma_i$, we see that

$$\lim_{i\to\infty} g_i \in \Gamma_\infty.$$

Recall from the theory that we wish to find

$$\lim_{i\to\infty} \gamma_i^* = q.$$

In $L_{1/7}$ a non-periodic infinite sequence of pre-images of any gap, be it finite sided or otherwise, tends to a point on S^1. Therefore, we have that

$$\lim_{i\to\infty} g_i = \lim_{i\to\infty} \gamma_i^*$$

meaning that

$$q \in \Gamma_\infty.$$

In practise we only need a finite portion of the tableau as the word generated by the tableau becomes periodic. Lemma 3.1.1 gives us a rough mechanism for deciding whether we have calculated enough of the tableau to compute μ_q.

Lemma 3.1.1 *If row i_1 is equal to row i_2 (that is,*

$$x_{i_1,(r_{i_1}+j)} = x_{i_2,(r_{i_2}+j)} \quad \forall j \geq 0)$$

for any $i_1 < i_2$ then Γ_∞ must be periodic of period n_q starting from letter r_{i_1}.

Proof: The conditions outlined in the lemma give that the rows of the tableau will be periodic of period $n|(i_2 - i_1)$ for rows $i \geq i_1$.

Once the rows of the tableau become periodic at column r_{i_1} we must have that Γ_∞ is periodic from the $r_{i_1}^{\text{th}}$ letter. We know that μ_q has period $n_q|n_p$. Because Γ_∞ labels a limit of triangles which converge on *one* endpoint of μ_q, of period n_q for $q \neq 1/3$, Γ_∞ must also have period n_q. ■

Hence, as soon as two equal rows of the tableau are calculated lemma 3.1.1 gives that the vague word will be periodic from this point on, meaning no more of the tableau is needed. As rows of the tableau become periodic after a reasonable number of pull backs (see chapter 5) this becomes a useful indicator for when no more of the tableau is needed.

3.2 Examples of the algorithm

3.2.1 The example of $p = 7/31$

The minor leaf $\mu_{7/31}$ connects 7/31 to 8/31. We look for the vertex closest to the orbit of $\mu_{7/31}$ among those of –T. The vertex 3/14 (outside labelling

11/14) is closest to the third forward image of μ_{7o31}, $\ell_{25/31,2/31}$, which connects the points with inside labelling 29/31 and 6/31.

We form the top row of the tableau by pulling back 3/14 close to the periodic pre-images of $\ell_{25/31,2/31}$ and labelling the pull backs according to figure 3.8. This generates the word

$$(BCL_1R_2R_3L_2)^\infty BCL_1R_2C.$$

The subwords beginning with $x_{0,3} = L_1$ and $x_{0,2} = R_2$ (that is, the words L_1R_2C and R_2C) denote infinite sided gaps which share vertices with T, meaning that their columns are labelled $c_3 = c_2 = L$. The subword beginning with $x_{0,5} = L_2$ labels an infinite sided gap which will share a vertex with $-T$, meaning that $c_5 = R$. As $c_i \neq R$ for $i < 4$, and $x_{0,4} = BC$ we have that $c_4 = BC$.

$$\begin{array}{c|ccccc|ccccc|cccc}
\ldots & BC & L_1 & R_2 & R_3 & L_2 & BC & L_1 & R_2 & R_3 & L_2 & BC & L_1 & R_2 & C. \\
 & & & & & & & & & & |-T & & |T & |T & \\
 & & & & & & & & & & & & & & \\
 & & & & & & & & & & R & BC & L & L &
\end{array}$$

Here the vertical lines separating the entries in the tableau are to imply the period. They occur between columns in which the top row approximates the major leaves and where the top row approximates the minor leaves. The lines below the elements in the tableau indicate leaves. Here only the leaves which are sides of T or $-T$ are shown. The R, BC and two Ls underneath the tableau are the column labels.

As there are not yet enough complete columns in the tableau to check for periodicity we must add at least one more row.

$$\begin{array}{c|ccccc|cccccc|cccc}
\ldots & BC & L_1 & R_2 & R_3 & L_2 & BC & L_1 & R_2 & & R_3 & L_2 & BC & L_1 & R_2 & C. \\
 & | & | & | & | & | & | & | & | & & | & |-T & & |T & |T & \\
\ldots & BC & L_1 & R_2 & R_3 & L_2 & BC & L_1 & R_1 & & R_2 & C & & & & \\
 & & & & & & & & |-T & & |T & & & & & \\
 & & & & & & & & & & & & & & & \\
 & & & & & & & & R & & L & R & BC & L & L &
\end{array}$$

Again, as there are only six complete columns (as the $-T$ in the seventh column can extend down to a new row) the tableau is not full enough to find q. Adding more rows as above yields the tableau shown in table 3.1.

As the second and fifth rows are equal lemma 3.1.1 gives that no more of the tableau is needed.

Then, the vague word resulting from the tableau is

$$(LRRLR)^{\infty}BCLL.$$

The trailing $BCLL$ is of no consequence and so we translate the periodic section of the vague word into the full word

$$(L_2R_2R_1L_2R_1)^{\infty}.$$

The word $L_2R_2R_1L_2R_1$ labels the interval $[93/224, 95/224)$ (with inside labelling). This contains only one point of period 5, 18/31 (in the outside labelling) gives the result

$$\mu_q = \ell_{13/31,18/31}.$$

$\dots$	L_2	BC	L_1	R_2	R_3	L_2	BC	L_1	R_2	R_3	L_2	BC	L_1	R_2	$C.$
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$		$\vert T$	$\vert T$	
$\dots$	L_2	BC	L_1	R_2	R_3	L_2	BC	L_1	R_1	R_2	C				
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$	$\vert T$					
$\dots$	L_2	BC	L_1	R_1	R_1	R_2	R_3	L_2	C						
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$							
$\dots$	L_2	BC	L_1	R_1	R_1	R_1	R_2	C							
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$	$\vert T$								
$\dots$	L_2	BC	L_1	R_1	R_2	C									
	$\vert$	$\vert$	$\vert$	$\vert -T$	$\vert T$										
$\dots$	R_2	R_3	L_2	C											
	$\vert$	$\vert$	$\vert -T$												
$\dots$	R_1	R_2	C												
	$\vert -T$	$\vert T$													
	R	L	R	R	L	R	L	R	R	L	R	BC	L	L	

Table 3.1: A section of the tableau for $\mu_{7/31}$

3.2.2 The example of $p = 10/63$

The minor leaf $\mu_{10/63}$ connects $10/63$ to $17/63$. The vertex of $-T$ which is closest to the orbit of $\mu_{10/63}$ is $5/14$ (outer labelling $9/14$). This vertex approximates the second forward image of $\mu_{10/63}$, $\ell_{40/63,5,63}$. Pulling back $5/14$ to form the top row of the tableau we get

$$(BCL_1R_2|R_3L_3L_2)^\infty$$

(here, the vertical bar occurs immediately before the column approximating the minor leaf).

The point giving $x_{0,1} = L_2$ is a vertex of $-T$ meaning that $c_1 = R$.

$$\begin{array}{l|cccccc|ccc}
\ldots & R_3 & L_3 & L_2 & BC & L_1 & R_2 & R_3 & L_3 & L_2 \\
 & & & & & & & & & \mid -T \\
 & & & & & & & & & \\
 & & & & & & & & & R
\end{array}$$

This leaf of $-T$ connects to the gap C. Adding a row, as before, gives

$$\begin{array}{l|cccccc|ccc}
\ldots & R_3 & L_3 & L_2 & BC & L_1 & R_2 & R_3 & L_3 & L_2 \\
 & \mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid -T \\
\ldots & R_3 & L_3 & L_2 & BC & L_1 & R_2 & R_3 & L_2 & C \\
 & & & & & & & & \mid -T & \\
 & & & & & & & & & \\
 & & & & & & & & R & R
\end{array}$$

Continuing in this way gives the tableau shown in table 3.2. The tableau in table 3.2 gives the vague word

$$(LUCLLRR)^\infty LRR.$$

Stripping the trailing, non-periodic, "LRR" leaves the periodic word $(LUCLLRR)^\infty$. This vague word translates into

$$(L_2UCL_1L_2R_2R_1)^\infty.$$

The word $L_2UCL_1L_2R_2R_1$ labels the region $[247/448, 249/448)$ (with inside labelling) on S^1 which contains the single period six point (with outside labelling) $q = 4/9$. Hence,

$$\mu_q = \ell_{4/9,5/9}.$$

…	R_2	R_3	L_3	L_2	BC	L_1	R_2	R_3	L_3	L_2	BC	L_1	R_2	R_3	L_3	L_2
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$
…	R_2	R_3	L_3	L_2	BC	L_1	R_2	R_3	L_3	L_2	BC	L_1	R_2	R_3	L_2	C
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$	
…	R_2	R_3	L_3	L_2	BC	L_1	R_2	R_3	L_3	L_2	BC	L_1	R_1	R_2	C	
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$	$\vert T$		
…	R_2	R_3	L_3	L_2	BC	L_1	R_2	R_3	L_3	L_2	R_3	L_2	C			
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$				
…	R_2	R_3	L_3	L_2	BC	L_1	R_1	R_2	UC	L_1	R_2	C				
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$	$\vert T$		$\vert T$	$\vert T$					
…	R_2	R_3	L_3	L_2	R_3	L_2	C									
	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert -T$										
…	R_1	R_2	UC	L_1	R_2	C										
	$\vert -T$	$\vert T$		$\vert T$	$\vert T$											
	R	L	UC	L	L	R	R	L	UC	L	L	R	R	L	R	R

Table 3.2: A section of the tableau for $\mu_{10/63}$

Chapter 4

The 'image' of the mating equivalence

Ben Wittner is responsible for providing us with much foundation for the work presented in this document - the Wittner flip mating between the aeroplane and the rabbit polynomials being the obvious example. Towards the end of [W] (chapter 11) two consequences of the theory presented therein are outlined which are distinctly relevant parallels to this work.

Let f be a mating where one of the constituent polynomials, f_0, lies in a limb L of the Mandelbrot set and the other, f_1, has 0, its critical point, as an eventually-fixed point. Complement 11.1.2 from [W] states that f must then be Thurston equivalent to a mating between the critically finite polynomial in the base component of L and some other polynomial, h_i, which is unique to f. Wittner then presents complement 11.1.3 which states that for a particular f_1, with critical point eventually fixed, and f_0 ranging over a particular L, all of the corresponding h_i of complement 11.1.2 lie in the same limb of the Mandelbrot set.

Mirroring complement 11.1.2 in this work is lemma 2.3.5, albeit restricted only to a particular subset of critically finite matings. Also as in complement 11.1.2 an algorithm for finding μ_q such that $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$ has already been presented. Theorem 4.0.1 continues in the pattern laid out in complement 11.1.3 of [W] and is the result worked towards in this section. Before presenting the theorem a definition is needed.

Definition 4.0.1 *The algorithm presented in chapter 3 produces a mating* $s_{1/7} \perp\!\!\!\perp s_q$ *from an initial mating of* $s_{3/7} \perp\!\!\!\perp s_p$*. Hence, this algorithm can be used* *to define a map*

$$p \mapsto q : X \to Y \subset [0, 1),$$

where X is the set of all odd denominator rationals in $(1/7, 2/7)$ *and Y is the* image *of the mating map.*

Theorem 4.0.1 *The image of the mating map, Y, satisfies*

$$Y \subset (1/3, 2/3).$$

The algorithm uses two sequences of triangles, Δ_i in the second invariant circle on $s_{3/7} \perp\!\!\!\perp s_p$, and δ_i in $L_{1/7}$, which correspond under the Thurston equivalence between $s_{3/7} \perp\!\!\!\perp s_p$ and $s_{1/7} \perp\!\!\!\perp s_q$. Theorem 3.0.6 states that the $\{\delta_i\}$ become arbitrarily close to an endpoint of μ_q. So, for large enough i, we may find a δ_i whose word shares a prefix of length n with the word labelling q (in the labelling of $L_{1/7}$, as is much of the symbolic dynamics in this chapter). If a vertex of the triangle δ_i has word $W = w_0 w_1 w_2 \ldots$ in the labelling of $L_{1/7}$ then a vertex of Δ_i must also have label W with respect to the labelling of $L_{1/7}$ on the second invariant circle (see section 3.1). We use the fact that Δ_i, for some i, must share a prefix with the word of q to deduce the possible values of q.

The labelling of Δ_i, with respect to $L_{1/7}$ on the second invariant circle, can be determined by how a path on the second invariant circle connects it to the periodic central gap of $L_{3/7}$. These paths are formalised as follows.

Definition 4.0.2 *A* connection *in the context of $L_{3/7} \cup L_p^{-1}$ is a path in the filled second invariant circle (described on page 32) which crosses S^1 a finite number of times and which only traverses one pre-image of the periodic central gap of $L_{3/7}$ of any given pre-period, which also does not cross the central periodic gap itself. Equivalently, a connection in the context of $L_{1/7}$ is a path on the unit disc crossing only finitely many leaves of $L_{1/7}$ (with the same restriction that it may only pass through one pre-image of the periodic central gap of any given pre-period) which does not cross the central periodic gap.*

Of the three gaps which bound a triangle in $L_{1/7}$, two are of equal pre-period and the other, which has the longest leaf of the triangle boundary in its boundary, is of a lower period. Hence, the restriction that a connection may only pass through one pre-image of the central periodic gap of any given pre-period means that, on $L_{1/7}$, a path connecting one gap to another must either always 'head towards' C or always 'head away' from C; it cannot change direction.

In chapter 3 we labelled gaps of $L_{3/7}$, which share a vertex with a Δ_i, with a word of $L_{1/7}$, consisting of L_1, L_2, R_1, R_2, UC, BC and C, with respect to the second invariant circle.

Lemma 4.0.2 *Given two gaps, G_1 and G_2, in $L_{1/7}$ with G_2 bounding G_1 from zero there exists a connection between G_1 and G_2.*

Proof: We wish to show that a path from one gap to another can be homotoped to cross only finitely many leaves of $L_{1/7}$ and that the path only crosses one pre-image of the central periodic gap of any given pre-period.

The only reason such a path would have to pass through infinitely many gaps is if there is an accumulation of leaves separating G_1 from G_2. There are no accumulation leaves in $L_{1/7}$, however. If there were then $\mu_{1/7}$ would be an accumulation leaf, contradicting the fact that it lies on the boundary of both a finite sided gap and a periodic infinite sided gap. Hence, it is possible to homotope any path on $L_{1/7}$ to pass through only finitely many gaps.

It is clear that the path need not pass through two gaps of the same pre-periodicity as $G_2 > G_1$. ■

In section 3.1, page 41, we saw how any word in the labelling of $L_{3/7}$ specifies a particular pre-image of the central periodic gap. Similarly, all words in the labelling of $L_{1/7}$ specify either the whole, or part of the boundary of a pre-image of the periodic central gap of $L_{1/7}$.

To see this we again append letters onto words so that they end with C. If the word, W, ends with L_1 or R_1 append L_2C so that $W' = WL_2C$. If W ends with L_2 or R_2 simply append C to form W'. If W ends with UC or BC then append L_1L_2C. The region of S^1 labelled by W' has one or two connected components and the boundary points of these components will be endpoints of pre-images of the two major leaves. Any pre-image of the major leaves may only lie in the boundary of one infinite sided gap in $L_{1/7}$ giving that the word

W' specifies a particular pre-image of the central gap. We associate this gap with W.

Lemma 4.0.3 *Assume G_1 is a gap of $L_{1/7}$ with label W and that W has prefix V. Then there exists a connection between G_1 and G_2, the gap associated with V.*

Proof: The gap G_2 bounds all points in the region labelled by V from zero, except possibly a countable set if G_2 is the periodic central gap. As the region labelled by W is a subset of that labelled by V we see that G_1 is either equal to G_2, in which case there is a trivial connection between G_1 and G_2, or G_1 is bounded from zero by G_2. Lemma 4.0.2 then gives the result. ■

In chapter 3 we see that for an odd denominator rational, q, to lie in Y it must be the limit of a sequence of triangles, $\{\delta_{m_i}\}$, in $L_{1/7} \cup L_q^{-1}$ (theorem 3.0.6). Here δ_i corresponds to $\Delta_i \subset L_{3/7} \cup L_p^{-1}$ according to the one to one correspondence of triangles discussed in the same chapter.

Any δ_i can be uniquely labelled by a word in $L_{1/7}$. To do this note that every non-periodic triangle in $L_{1/7}$ is a pre-image of $-T$. Label each triangle in the forward itinerary of δ_i, under $s_{1/7}$, by the region containing all of its vertices, ending by labelling $-T$ by C.

Corollary 4.0.4 *If the odd denominator rational, q, in $L_{1/7}$ is labelled by a word with prefix word W then, in $L_{3/7}$, there must exist a connection, which does not traverse the central gap of $L_{3/7}$, between the gap of $L_{3/7}$ with label W, with respect to the second invariant circle, and a point on the second invariant circle which lies in the interval $(5/7, 6/7)$ on S^1.*

Proof: As the δ_i approximate μ_q there must exist a δ_{i_0} within any arbitrary neighbourhood of μ_q. A neighbourhood of μ_q can be chosen so that the triangle δ_{i_0} has label prefixed by W. Now, δ_{i_0} in $L_{1/7}$ corresponds to Δ_{i_0} in $L_{3/7} \cup L_p$, also with label prefixed by W (with respect to the labelling of $L_{1/7}$ on the second invariant circle). As Δ_{i_0} is approximating μ_p, providing i_0 is large enough, Δ_{i_0} will have vertices which lie in the interval $(5/7, 6/7)$ on S^1 (or $(1/7, 2/7)$ in the outside labelling).

Lemma 4.0.3 shows that there exists a connection between Δ_{i_0} and the gap in $L_{3/7}$ with word W. ■

Given that we know certain connections must exist for a given q to lie in Y we illustrate a mechanism which may prevent such connections existing.

Take a sequence of triangles on the exterior of S^1 in $L_{3/7} \cup L_p^{-1}$, $\{d_i\}$ say, to be constructed so that $d_0 = T$, $d_1 = -T$ and, for $i \geq 1$, d_{i+1} is the pre-image of d_i such that all vertices of d_{i+1} are bounded from ∞, in the exterior of the closed unit disk, by d_i (see figure 4.1).

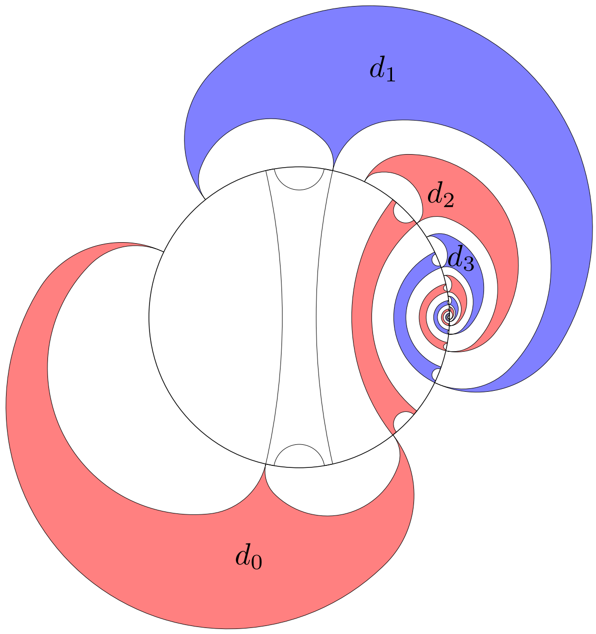

Figure 4.1: The sequence $\{d_o\}$ and joining gaps.

From the construction, d_0 is connected to d_2 by the gap attached to the vertex of $d_0 = T$ at 6/7. As d_2 is connected to T we see that d_2 lies in the region labelled L in $L_{1/7}$. Also, d_1 is connected to $d_1 = -T$ by a pre-image of this gap. In fact, the gap connecting d_0 to d_2 pulls back to connect all d_i to d_{i+2}. This means that for i even d_i is connected to T and for i odd d_i is connected to $-T$. This gives rise to the situation depicted in figure 4.2.

The $\{d_{2i}\}$ together with the gaps of $L_{3/7}$ connecting them together form a continuous spiral homing in on the point on S^1 with angle of rotation 0.

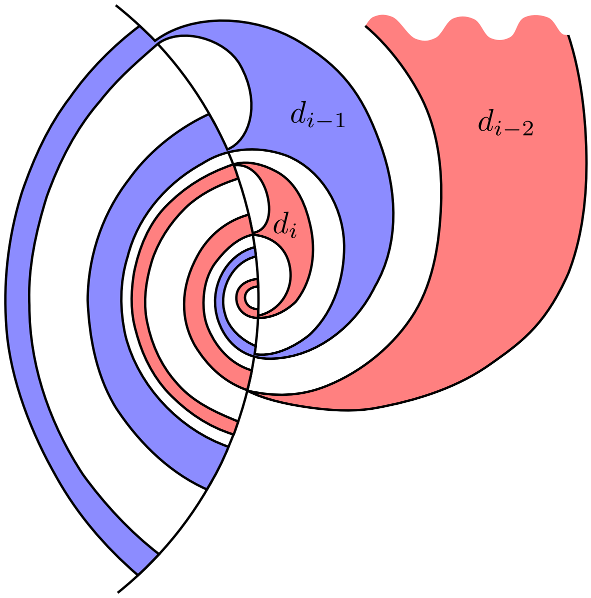

Figure 4.2: An obstruction at 0. Triangle d_{i-1} is a pre-image of d_{i-2} giving that the spiral of triangles and gaps is infinite.

Likewise, the $\{d_{2i+1}\}$ together with the gaps of $L_{3/7}$ which connect them form another continuous spiral homing in on the point with angle of rotation 0. Take the union of these two spirals together with the central periodic gap of $L_{3/7}$ (joining d_0 to d_1). The limit of this set separates the plane into two connected components (see figure 4.3).

As a connection in $L_{3/7} \cup L_p^{-1}$ crosses S^1 only finitely many times it may not pass between these two infinite spirals at 0. Hence, nothing on one side of the obstruction may be connected to anything on the other. In particular, a triangle, Δ, in $L_{3/7} \cup L_p^{-1}$ on the side of the obstruction containing the interval $(5/7, 6/7)$ of S^1 cannot be connected to gaps on the other side of the obstruction. Then, consider a gap in $L_{3/7}$ in the full orbit of the central periodic gap which is associated with the word W. If this gap lies on the opposite side of the obstruction to the interval $(5/7, 6/7)$ of S^1 then no such Δ may be labelled (relative to the labelling of $L_{1/7}$ on the second invariant circle) by a word prefixed by W. This gives, as in the result of corollary 4.0.4 that q, also, cannot be prefixed by such a W.

Note that the obstruction is caused by a sequence of triangles, $\{d_i\}$, which

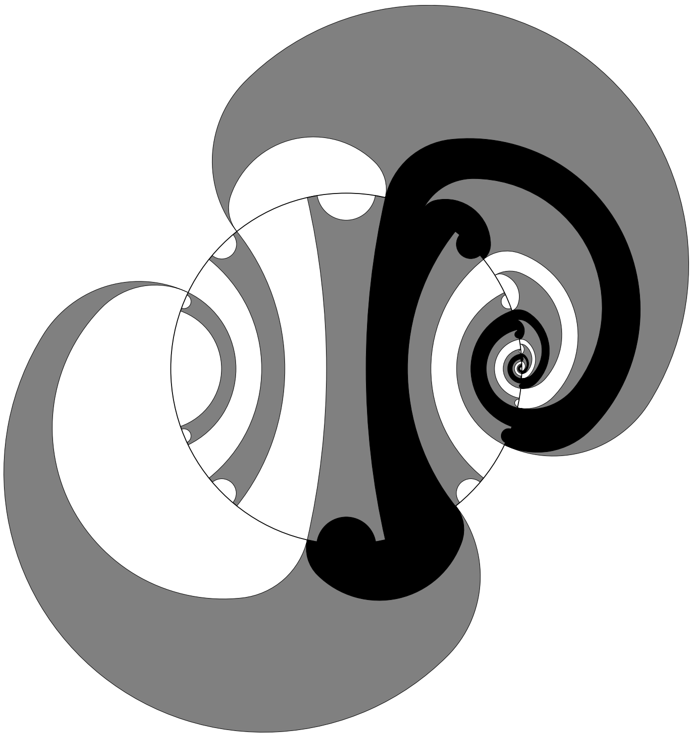

Figure 4.3: The obstruction (grey) separating the plane (black and white).

pull back to approximate zero. Recalling that, in $L_{1/7}$, the triangle d_{2n} lies in L and d_{2n+1} lies in R we see that d_{2n} is labelled by the word $(L_2R_1)^n$ and d_{2n+1} has word $R_1(L_2R_1)^n$. In $L_{1/7}$, taking the limit as $n \to \infty$, these words label 1/3 and 2/3, respectively.

Lemma 4.0.5 *Any gap in the second invariant circle on $L_{3/7} \cup L_p^{-1}$ which has a label with prefix in*

$$\begin{array}{l} UC, \\ L_1, \\ L_2(R_1L_2)^n BC, \\ (R_1L_2)^n R_2, \\ (R_1L_2)^n UC, \\ R_2. \end{array}$$

lies on the opposite side of the obstruction to the interval $(5/7, 6/7)$ on S^1.

Proof: Figure 4.4 shows that the gaps with label UC, R_2 and L_1 are

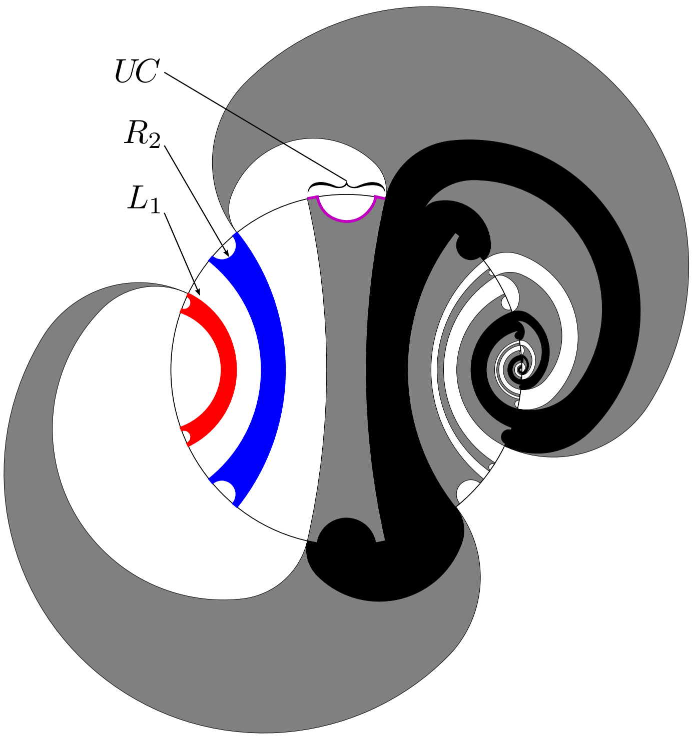

Figure 4.4: The second invariant circle with gaps with label UC (one end of the central gap), L_1 (a whole gap) and R_1 (a whole gap) highlighted.

clearly on the opposite side of the obstruction to the interval $(5/7, 6/7)$ on S^1. Figure 4.5 highlights the gaps $L_2(R_1L_2)^nBC$, $(R_1L_2)^nR_2$ and $(R_1L_2)^nUC$ for

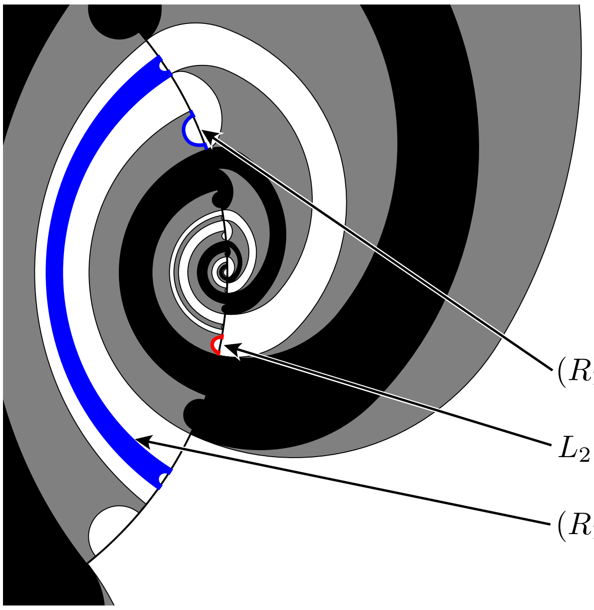

Figure 4.5: A section of the second invariant circle with gaps with label $(R_1L_2)^nUC$ (one end of a gap), $L_2(R_1L_2)^nBC$ (one end of a gap) and $(R_1L_2)^nR_2$ (a whole gap) highlighted, for $n = 1$.

$n = 1$. For this value of n they evidently lie on the opposite side of the obstruction to the interval $(5/7, 6/7)$ of S^1. However, for higher values of (and indeed, lower values of n in the case of L_2BC) the picture is much the same. While the scale of the objects involved changes the configuration does not. ∎

Proof: (of theorem 4.0.1) Figure 4.6 shows that all points in $[0, 1/3) \cup (2/3, 1]$ are labelled by a word with prefix in

$$\begin{array}{l} UC, \\ L_1, \\ L_2(R_1L_2)^nBC, \\ (R_1L_2)^nR_2, \\ (R_1L_2)^nUC, \\ R_2. \end{array}$$

Lemma 4.0.5 shows that, on the second invariant circle on $L_{3/7} \cup L_p^{-1}$, the points on S^1 with such labels lie on the opposite side of the obstruction to p. Hence, there exists no connection between any point with angle in $[0, 1/3) \cup (2/3, 1]$ on the second invariant circle and p, meaning that q must lie in $(1/3, 2/3)$. ■

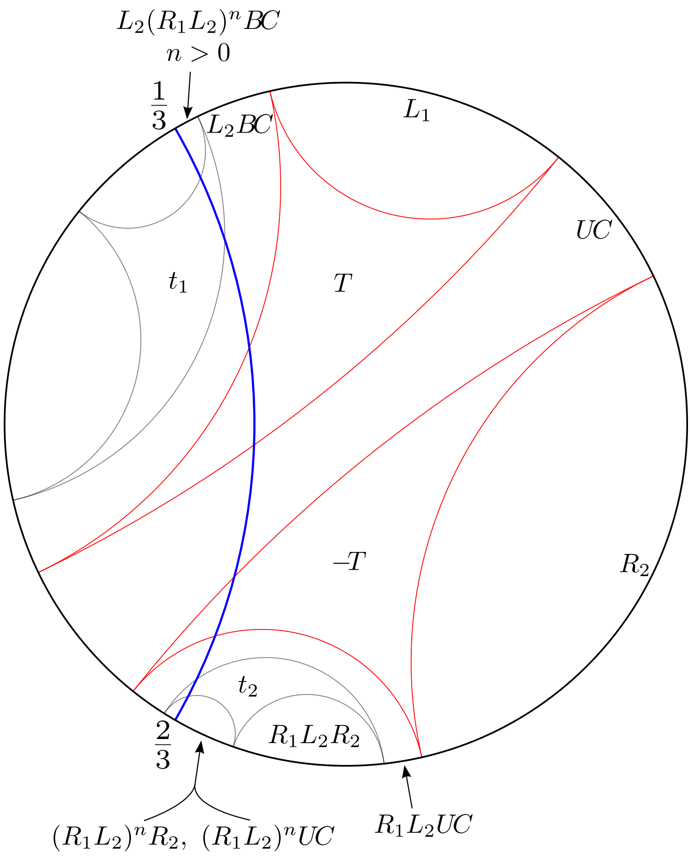

Figure 4.6: Symbolic labelling of regions of S^1 in $L_{1/7}$. The region between $2/3$ and the middle vertex of t_2 is made up of regions alternately labelled $(R_1L_2)^nR_2$ and $(R_1L_2)^nUC$.

4.0.3 Equivalent minor leaves on the QML

Figure 4.7 shows Thurston's quadratic minor lamination with all periodic leaves of period less than or equal to 14. All leaves in the domain of the equivalence map (all periodic leaves greater than $\mu_{1/7}$) are highlighted in blue and the image of these leaves, in Y, is highlighted in red.

The range, Y, can clearly be seen to be greater than $\mu_{1/3}$, as required by

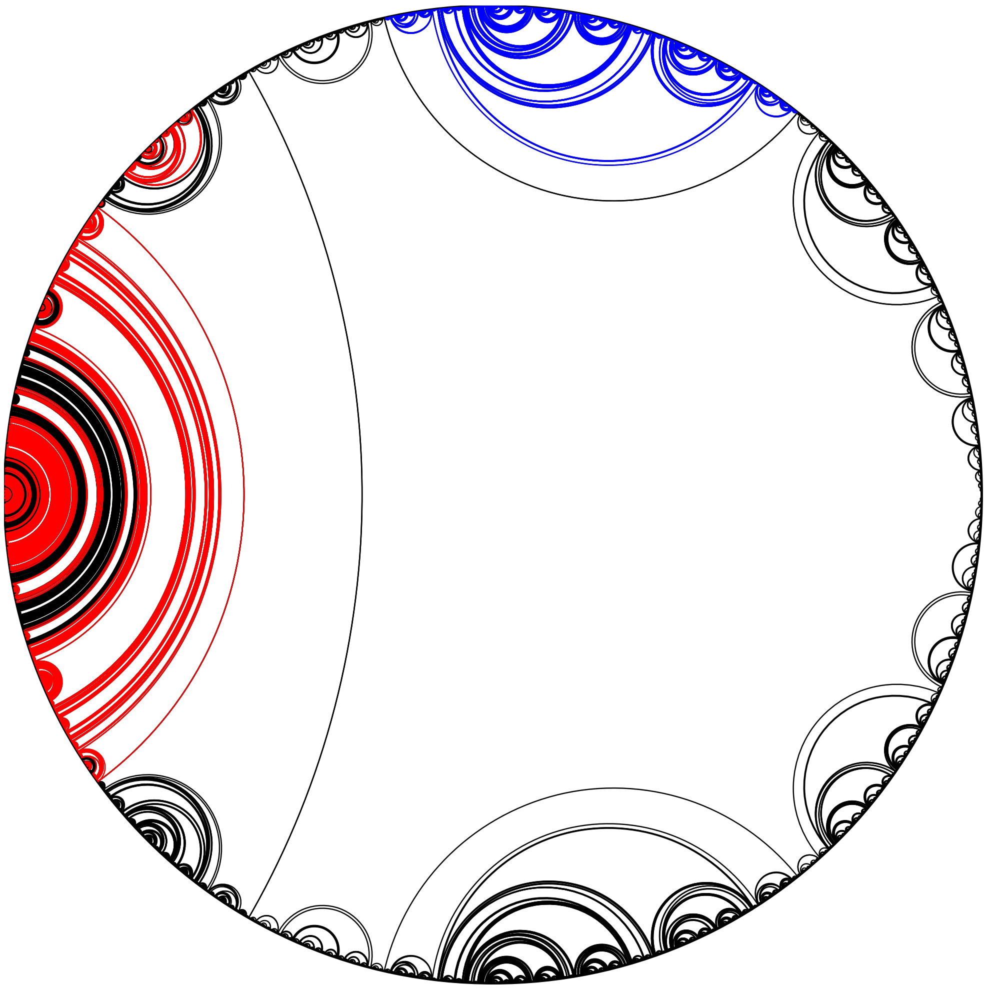

Figure 4.7: The QML, including periodic leaves of period up to and including 14, with μ_p highlighted in blue and μ_q highlighted in red where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$ for some $\mu_p > \mu_{1/7}$.

theorem 4.0.1. Other properties of Y can be hypothysised from the figure. It appears that there are open intervals in $(1/3, 2/3)$ which contain no leaves of Y (for example $(3/5, 2/3)$ and $(1/3, 364/1023)$). This appears reasonable as, by way of the obstruction, we have already seen that an open interval, $[0, 1/3) \cup (2/3, 1]$, does not intersect Y and there is no reason to believe there are no other obstructions. It seems reasonable to expect that there exists no open interval for which all contained odd denominator rationals lie in Y (although this is trivially incorrect for leaves of any fixed range of periods).

Chapter 5

Convergence of the Algorithm

In this section we consider the tableau formed by the algorithm and determine an upper bound on the number of pull backs required for the algorithm to converge for a minor leaf of a given period.

5.1 Families of Examples

To motivate the general case we look at families of symbolic words from our labelling of the aeroplane ($L_{3/7}$). Taking the minor leaves which are specified by each member of the family we are able to take advantage of properties of the laminations of the families to quickly calculate their equivalent minor leaf through the usual equivalence.

As our minor leaves must be greater than $\mu_{1/7}$, taking into account that the rabbit lamination is on the exterior of the unit disk, all members of any families we consider begin with BC or R_3.

The following definition proves key to this section.

Definition 5.1.1 *A* colour change*, in the context of a column of a tableau, refers to two adjacent letters in the column one of which being L_2 and the other being R_2. The name comes from assigning the colour red to the labels L_1, L_2, and L_3 and the colour blue to the labels R_1, R_2, and R_3.*

The examples that are examined will be described by iterative functions which each track a property of the tableau. The significance of the functions we choose stems from the fact that many of the key properties of the tableaux between major leaf columns can be prescribed by the bottom-most, non-C, entry in the right-most major leaf column. Indeed this entry dictates the the tableau completely in some examples.

5.1.1 The Family of $BCL_1R_1^kR_2R_3L_2$

We begin by describing a number of functions particular to this example.

Definition 5.1.2 *Define $t(n)$ to be the number of colour changes in the n^{th} major column (n^{th} from the right side of the tableau).*

While we see the formula for $t(n)$ in lemma 5.1.2 a number of other functions must first be explained.

Definition 5.1.3 *Let b_n denote the bottom most, non-C letter of the n^{th} major leaf column. Then define the function $b(X, \sigma)$, with X a symbolic letter of $L_{3/7}$ so that*

$$b_n = b(b_{n-1}, \sigma)$$

where σ is the parity of $t(n)$.

An advantage of restricting consideration to individual families of examples is that we may give sharp bounds on the number of steps required for the tableau to converge.

Theorem 5.1.1 *For any family parameter k there exists an N such that $\forall i, j > N$ we have that $b_i = b_j$. Further,*

$$N = \log_2 k + 2.$$

Definition 5.1.4 *Consider a given major leaf column, c. As the tableaux extend downwards there are potentially rows in this column which are not present in the previous major column. Write $s(X)$ for the count of colour changes present between these rows, or between the upper-most of these rows and the row above, in c.*

To calculate $s(X)$ and $b(X, \sigma)$ it is necessary to calculate all of the possible blocks of the tableau that may be added between one major column and the next.

In this example we may use $BCL_1R_1^kR_2R_3L_2$ as the top row of the tableau.

$$
\begin{array}{c|cccccccc}
X_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_2 & R_3 & L_2 \\
| & | & | & | & | & | & | & | & |-T \\
X_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_1 & R_2 & C \\
| & | & | & | & | & | & |-T & |T & \\
Y_2 & BC & L_1 & R_1^{k-2} & R_1 & R_2 & C & & \\
| & | & | & | & |-T & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

Depending on whether k is even or odd there can be one of two configurations for the bottom of these n-columns of the tableau.

For k odd

$$
\begin{array}{c|ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & R_1 & \ldots \\
| & | & | & | & |-T & \\
X_2 & BC & L_1 & R_2 & C & \\
 & & |T & |T & &
\end{array}
$$

$b(L_2, \sigma) = X_2.$

For k even

$$
\begin{array}{c|cccc}
\vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & \ldots \\
| & | & | & |-T & \\
X_2 & R_3 & L_2 & C & \\
| & | & |-T & & \\
X_1 & R_2 & C & & \\
 & |T & & &
\end{array}
$$

$b(L_2, \sigma) = X_1.$

The formula for s is

$$s(L_2) = 1 + \left\lfloor \frac{k}{2} \right\rfloor .$$

Next we consider a section of the tableau to find $b(R_1, \sigma)$. The case where $b_i = L_2$ illustrated how a $b_i = R_1$ could arise: we use a continuation of the case $b_i = L_2$ where k is even to arrive at $b_{i+1} = R_1$ and calculate $b_{i+2} = b(R_1, \sigma)$.

$$
\begin{array}{c|cccccccc|c}
X_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_1 & R_1 & R_2 & R_3 \ldots \\
| & | & | & | & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_1 & R_1 & R_1 & R_2 \ldots \\
| & | & | & | & | & | & | & | & |-T & \\
X_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_1 & R_2 & C & \\
| & | & | & | & | & | & |-T & & & \\
Y_2 & BC & L_1 & R_1^{k-2} & R_1 & R_2 & C & & & \\
| & | & | & | & |-T & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & &
\end{array}
$$

We complete the bottom of this section of tableau according to the parity of k.

For k odd

$$
\begin{array}{c|lllll}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & R_1 & \dots \\
| & | & | & | & |-T & \\
X_2 & BC & L_1 & R_2 & C & \\
 & & |T & |T & &
\end{array}
$$

$b(R_1, \sigma) = X_2$.

For k even

$$
\begin{array}{c|llll}
\vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & \dots \\
| & | & | & |-T & \\
X_2 & R_3 & L_2 & C & \\
| & | & |-T & & \\
X_1 & R_2 & C & & \\
 & |T & & &
\end{array}
$$

$b(R_1, \sigma) = X_1$.

The formula for s is

$$s(R_1) = 1 + \left\lfloor \frac{k}{2} \right\rfloor .$$

Next we extend backwards to find $b(L_1, \sigma)$ and $s(L_1)$.

$$
\begin{array}{c|llllllllll|l}
X_2 & BC & L_1 & R_1^{k-3} & R_1 & R_1 & R_1 & R_2 & R_3 & L_2 & R_3 \dots \\
| & | & | & | & | & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k-3} & R_1 & R_1 & R_1 & R_2 & BC & L_1 & R_2 \dots \\
| & | & | & | & | & | & |-T & & & & \\
Y_2 & BC & L_1 & R_1^{k-3} & R_1 & R_2 & C & & & & \\
| & | & | & | & |-T & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & & &
\end{array}
$$

Then

For k odd

$$
\begin{array}{c|llll}
\vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & \dots \\
| & | & | & |-T & \\
X_2 & R_3 & L_2 & C & \\
| & | & |-T & & \\
X_1 & R_2 & C & & \\
 & |T & & &
\end{array}
$$

$b(L_1, \sigma) = X_1$.

For k even

$$
\begin{array}{c|lllll}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & R_1 & \dots \\
| & | & | & | & |-T & \\
X_2 & BC & L_1 & R_2 & C & \\
 & & |T & |T & &
\end{array}
$$

$b(L_1, \sigma) = X_2$.

The formula for s is

$$s(L_1) = \left\lfloor \frac{k+1}{2} \right\rfloor .$$

Similarly to find $s(R_2)$ and $b(R_2,\sigma)$.

$$
\begin{array}{c|ccccccc|c}
X_2 & BC & L_1 & R_1^{k-1} & R_1 & R_2 & R_3 & L_2 & BC\ldots \\
| & | & | & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k-1} & R_1 & R_1 & R_1 & R_2 & R_2\ldots \\
| & | & | & | & | & | & |-T & & \\
Y_2 & BC & L_1 & R_1^{k-1} & R_1 & R_2 & C & & \\
| & | & | & | & |-T & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

Then

For k odd

$$
\begin{array}{c|cccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
Y_2 & BC & L_1 & R_1 & R_1\ \ldots \\
| & | & | & | & |-T \\
X_2 & BC & L_1 & R_2 & C \\
 & & |T & |T &
\end{array}
$$

$b(R_2,\sigma) = X_1.$

For k even

$$
\begin{array}{c|ccc}
\vdots & \vdots & \vdots & \vdots \\
Y_2 & BC & L_1 & R_1\ \ldots \\
| & | & | & |-T \\
X_2 & R_3 & L_2 & C \\
| & | & |-T & \\
X_1 & R_2 & C & \\
 & |T & &
\end{array}
$$

$b(R_2,\sigma) = X_2.$

The formula for s is

$$s(R_2) = 1 + \left\lfloor \frac{k+1}{2} \right\rfloor .$$

It remains to explain the role played by the σ argument in the b function. The top row of the tableau for any example in this class is $BCL_1R_1^kR_2R_3L_2$. The major column, excluding, at most, the bottom two rows of the tableau, will consist entirely of the letters R_2 and L_2. The function $t(n)$ gives the number of colour changes present in any given major column and we know the the top-most letter is L_2. Letting σ be the parity of $t(n)$ the function b may be fully described as

	k odd	k even
$b(L_2,\sigma)$	X_2	X_1
$b(L_1,\sigma)$	X_1	X_2
$b(R_2,\sigma)$	X_1	X_2
$b(R_1,\sigma)$	X_2	X_1

where

$$X = \begin{cases} L & \text{for } \sigma \text{ even} \\ R & \text{for } \sigma \text{ odd.} \end{cases}$$

Similarly for s.

$$\begin{aligned} s(L_2) = s(R_1) &= 1 + \left\lfloor \frac{k}{2} \right\rfloor \\ s(L_1) &= \left\lfloor \frac{k+1}{2} \right\rfloor \\ s(R_2) &= 1 + \left\lfloor \frac{k+1}{2} \right\rfloor . \end{aligned}$$

A connection between adjacent rows is a leaf. Without specific knowledge of our major leaves we may still deduce some properties of such a connection by taking into account the gaps T and $-T$ in L_p.

Take a row from which a leaf, ℓ, connects to a lower row in a given column and assume we have knowledge of all entries in the upper row. We may calculate the pull-backs of the leaf ℓ, and so entries in the lower row, up until it pulls back into what would be the central gap of $L_{1/7}$ (as it is only then that it could pull back long or short without crossing T or $-T$). This family of examples has only one forward image of the minor leaf in the central gap. Hence, if ℓ lies in a major column then the $n-1$ pull-backs required to complete the lower row up until the next major column are known (as there are no colour change columns other than the major columns). Further, if the endpoint of the leaf ℓ given by the upper row follows one endpoint of the periodic backward orbit of μ_p then the other endpoint of ℓ, if in R_2 or L_2, will follow an endpoint of the periodic backward orbit of μ_p also, the choice of which endpoint depending on whether ℓ is long or short.

Applying this knowledge inductively, taking into account that the top row *is* the orbit of the minor leaf, we see that any R_2 or L_2 in a major column pulls back as one endpoint of the minor leaf for at least $n-1$ pull backs if there are only R_2s and L_2s above it in the major column.

In the tableaux above the left-most major column has many entries with subscript equal to two. These elements will pull back as one of the endpoints of the minor leaf unless there is an element above them in the major column with subscript not equal to 2, as described above. The tableaux above show that the only way an element with subscript *not* equal to 2 can arise in the major column is as the bottom-most non-C element of the major column. The cases of this including both L_1 and R_1 are illustrated above and do not result in any subscript 2 element in the major column lying below any element of subscript 1. Hence, we have that all subscript 2 elements in the major column have no elements with subscript other than 2 above them, giving that they pull back as one endpoint of the minor leaf.

The values of $b(X, \sigma)$ and $s(X)$ can be computed simply by hand as they are only concerned with the rows in the tableau which were not present in the major column to the right of the major column under consideration. However, a whole column is involved with the computation of $t(n)$ and so both the new sections of the tableau, and the pre-existing rows must be taken into account to find a formula for its value.

Lemma 5.1.2 *For the examples of $BCL_1R_1^kR_2R_3L_2$, $k \geq 0$ the count of colour changes in the n^{th} major leaf column, $t(n)$, satisfies*

$$t(n) = \left\lfloor \frac{t(n-1)}{2} \right\rfloor + s(b_{n-1}).$$

Proof: The "$s(b_{n-1})$" term in the formula above accounts for all new colour changes which are added to the n^{th} major column in the rows which were not present in the $(n-1)^{\text{th}}$ major column. It remains to explain the remainder of the formula.

We are considering the example of $BCL_1R_1^kR_2R_3L_2$. Figure 5.1 shows the orbit for the minor leaf of period nine with endpoint $(BCL_1R_1^4R_2R_3L_2)^\infty$, along with the triangle $-T$.

The short blue edge leaf of $-T$ in figure 5.1 corresponds to an

$$\begin{array}{c} L_2 \\ | \\ C \end{array}$$

connection in the tableau. This leaf bounds no forward image of μ_p meaning it will never pull back long. Hence, any rows that pull back long must begin with an

$$\begin{array}{c} R_1 \\ | \\ C \end{array}$$

connection.

Further, the long blue leaf in figure 5.1, which corresponds to any

$$\begin{array}{c} R_1 \\ | \\ C \end{array}$$

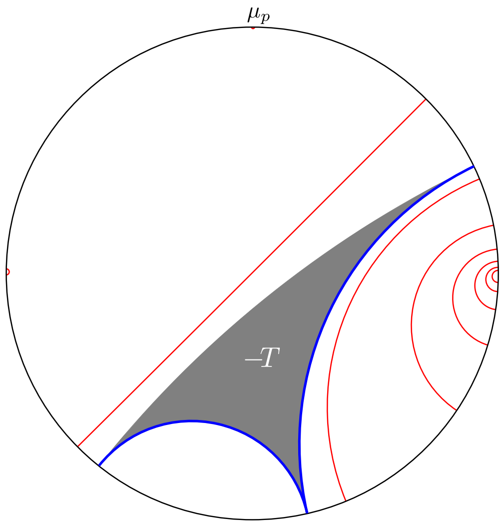

Figure 5.1: The orbit of μ_p, $p = 127/511$, in red. The two blue leaves of $-T$ are those corresponding to the $L_2 - C$ and $R_1 - C$ connections.

connection in the tableau, bounds all forward images of the minor leaf which are connected to the region labelled R_1. The forward images of the minor leaf which are connected to the region labelled R_1 lie in all columns of the tableau where the entry in the minor leaf row (that is, the top row) is not equal to BC, L_1, or L_2. Hence, if the right-most element in a row is an endpoint of such an $R_1 - C$ connection in a column whose element in the top row is not BC, L_1, or L_2 (that is, the major and minor columns and the column immediately following the minor column) then the leaf will pull back to bound the minor leaf. This gives that any pair of rows beginning with an

$$\begin{array}{c} R_1 \\ | \\ C \end{array}$$

connection in any column *not* headed by a BC, L_1, or L_2 must pull back long at the major column. Although the orbit of μ_p will change with k its basic configuration will not; the endpoints of the backward image of the side of $-T$ will arrive to approximate μ_p with the same ordering meaning that this argument holds for every member of this family.

Take ℓ to be any leaf which is represented by $R_1 - C$ in the tableau. Then ℓ is approximating a leaf, ℓ', in the forward orbit of the minor leaf. The endpoint of ℓ' nearest the endpoint of ℓ which lies in R_1 pulls back to a point in R_2 in the major column. Similarly the endpoint of ℓ' which is closest to the endpoint of ℓ in C pulls back to L_2 in the major column. Hence, if ℓ pulls back to an

$$\begin{array}{c} R_2 \\ | \\ L_2 \end{array}$$

connection in the major column then it has stayed close to the periodic orbit of the minor leaf; it has pulled back close to the periodic major leaf. This would give that is would continue to pull back long in subsequent major leaf columns. If, however, it pulls back to an

$$\begin{array}{c} L_2 \\ | \\ R_2 \end{array}$$

connection in the major column then it has pulled back closer to the non-periodic major leaf. This means that in the next major column it will *not* have pulled back long. Not only does this hold throughout the tableaux for this example, but for all of the three examples in this section.

The tableaux have top rows with L_2 in the major leaf position. Hence, if there are $t(n-1)$ colour changes in the column containing the $(n-1)^{\text{th}}$ occurrence of the pull back of μ_p (i.e. the major leaf) then every *second* one will be an

$$\begin{array}{c} R_2 \\ | \\ L_2 \end{array}$$

colour change. As it is only these "colour change" connections that pull back long again, exactly

$$\left\lfloor \frac{t(n-1)}{2} \right\rfloor$$

remain in the same rows in the following major leaf column. ■

It is now possible to examine the various functions and derive properties of the family of tableaux from them directly.

As

$$\begin{array}{ll} k \text{ odd } : & s(L_2) = s(L_1) = s(R_1) = \tfrac{k+1}{2}, \quad s(R_2) = \tfrac{k+3}{2}, \\ k \text{ even } : & s(L_2) = s(R_2) = s(R_1) = \tfrac{k+2}{2}, \quad s(L_1) = \tfrac{k}{2} \end{array}$$

we have that

$$t(n) = \begin{cases} \frac{t(n-1)-1}{2} + \frac{k+1}{2} = \frac{t(n-1)+k}{2} & t(n-1), k \text{ odd }, b_{n-1} = R_1 \\ \frac{t(n-1)-1}{2} + \frac{k+3}{2} = \frac{t(n-1)+k+2}{2} & t(n-1), k \text{ odd }, b_{n-1} = R_2 \\ \frac{t(n-1)}{2} + \frac{k+1}{2} = \frac{t(n-1)+k+1}{2} & t(n-1) \text{ even}, k \text{ odd} \\ \frac{t(n-1)-1}{2} + \frac{k+2}{2} = \frac{t(n-1)+k+1}{2} & t(n-1) \text{ odd }, k \text{ even} \\ \frac{t(n-1)}{2} + \frac{k}{2} = \frac{t(n-1)+k}{2} & t(n-1), k \text{ even}, b_{n-1} = L_1 \\ \frac{t(n-1)}{2} + \frac{k+2}{2} = \frac{t(n-1)+k+2}{2} & t(n-1), k \text{ even}, b_{n-1} = L_2. \end{cases}$$

From this we see that

$$\frac{k - t(n-1)}{2} \leq t(n) - t(n-1) \leq \frac{k + 2 - t(n-1)}{2} \tag{5.1}$$

which illustrates two things. Firstly, that $t(n) > t(n-1)$ while $t(n-1) < k$, and secondly, by adding $t(n-1)$ to the inequality, that $t(n-1) < k+2$ implies that $t(n) < k+2$, also. Hence, there exists some n such that $t(n) \geq k$ but $t(n)$ always satisfies

$$t(n) < k + 2.$$

Considering the cases of k odd and k even separately it is possible to state which values the functions t and b will converge to as n increases.

If $t(n-1) < k$ then

$$\begin{aligned} t(n) &< \frac{t(n-1) + k + 2}{2} \\ &< k + 1 \end{aligned}$$

meaning that $t(n) = k$ before $t(n+1) = k+1$.

Assume k odd. Take n to be the smallest integer such that $t(n-1) = k$. This gives that $b_{n-1} = R_1$ or R_2.

If $b_{n-1} = R_1$ then $t(n) = k$ and $b_n = R_2$. Then $t(n+1) = k+1$ and $b_{n+1} = L_1$ at which point the functions stabilise.

If $b_{n-1} = R_2$ then $t(n) = k+1$, $b_n = L_1$ at which point the functions stabilise, as for the case of $b_{n-1} = R_1$.

Assume k even and take n to be the smallest integer such that $t(n-1) = k$. Then $b_{n-1} = L_1$ or L_2.

If $b_{n-1} = L_1$ we have that $t(n) = k$, $b_n = L_2$. It follows that $t(n+1) = k+1$ and $b_{n+1} = R_1$ at which point the functions stabilise.

If $b_{n-1} = L_2$ then $t(n) = k+1$, $b_n = R_1$. It follows that $t(n+1) = k+1$ and $b_{n+1} = R_1$ at which point the functions stabilise.

So for any example in this family the word, and hence, mating, which results from the tableau can be given by the lower section of the tableau which results from one of two values of b_n. That is

$$\begin{array}{c} k \text{ odd} \quad \Rightarrow \exists N : b_n = L_1 \quad \forall n > N, \\ (\Rightarrow \text{ word } = L_2R_2R_1(L_2R_1)^{\frac{k-1}{2}}L_2BCL_1) \\ k \text{ even} \quad \Rightarrow \exists N : b_n = R_1 \quad \forall n > N. \\ (\Rightarrow \text{ word } = L_2R_2(R_1L_2)^{\frac{k}{2}}R_1L_2R_1) \end{array}$$

Now we derive the bound for N.

Proof: (of theorem 5.1.1) While $t(n) \neq t(n-1)$ we have that $b(n)$ does not necessarily equal $b(n-1)$. Hence, we wish to find N such that $t(n) = t(n-1)\ \forall n > N$

Because $t(n) > t(n-1)$ while $t(n-1) < k$ we wish to calculate the greatest possible number of steps taken for $t(n)$ to become equal to k

We see from equation 5.1 that

$$t(n) \geq \frac{t(n-1)+k}{2},$$

and $t(1) \geq k/2$. This gives that

$$t(n) \geq \frac{(2^n-1)k}{2^n}$$

from which we may deduce N.

We require

$$\begin{array}{rrcl} & t(n) & > & k-1 \\ \iff & \dfrac{(2^n-1)k}{2^n} & > & k-1 \\ & 1+k-\dfrac{k}{2^n} & > & k \\ & 1 & > & \dfrac{k}{2^n} \\ & 2^n & > & k \\ & n & > & \log_2 k. \end{array}$$

Hence, for all $n > \log_2 k$ we have that $t(n) \geq k$. Looking back to the work above we see that if $t(n) = k$ then $t(i) = t(j)\ \forall i, j \geq n + 2$.

Hence, taking

$$N = \log_2 k + 2$$

gives the desired result. ■

This shows that for any minor leaf in this family, the corresponding tableau will converge before

$$\text{period}(\mu_p).N = \text{period}(\mu_p).(\log_2 k + 2)$$

pull backs have been taken.

5.1.2 The Family of $BCL_1R_1^kR_2R_3L_2R_3L_2R_3L_2$

This subsection begins in much the same way as the last. The definition of $t(n)$, $b_n = b(X, \sigma)$, and $s(X)$ for this example read exactly as definitions 5.1.2, 5.1.3, and 5.1.4. However, as the tableau from which they are derived has a different form the values taken by them will unsurprisingly differ.

The following theorem bounds the number of steps involved in the algorithm for this family.

Theorem 5.1.3 *There exists an N such that for any k and $\forall i, j > N$ we have that $b_i = b_j$. Further,*

$$N = \log_2 k + 2.$$

The values of $b(X, \sigma)$ and $s(X)$ must be again computed by hand for this example. It is worth noting that in section 5.1.1 all example tableaux ended with either

$$
\begin{array}{c|llll}
\vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & \ldots \\
| & | & | & |-T & \\
X_2 & R_3 & L_2 & C & \\
| & | & |-T & & \\
X_1 & R_2 & C & & \\
 & |T & & &
\end{array}
$$

which will now be referred to as *case* α, or

$$
\begin{array}{c|lllll}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & R_1 & \ldots \\
| & | & | & | & |{-T} & \\
X_2 & BC & L_1 & R_2 & C & \\
 & & |T & |T & &
\end{array}
$$

which will be refereed to as *case* β. All tableaux in this section end in one of these cases also. Because of this the labels α and β will be used instead of reproducing the sections of tableau *ad nauseum*.

As before, we consider a section of the tableau to find $b(L_2, \sigma)$.

$$
\begin{array}{c|lllllllllll}
X_2 & BC & L_1 & R_1^{k-1} & R_1 & R_2 & R_3 & L_2 & R_3 & L_2 & R_3 & L_2 \\
| & | & | & | & | & | & | & | & | & | & | & |{-T} \\
X_2 & BC & L_1 & R_1^{k-1} & R_1 & R_2 & R_3 & L_2 & BC & L_1 & R_2 & C \\
| & | & | & | & | & | & | & |{-T} & & & \\
X_2 & BC & L_1 & R_1^{k-1} & R_1 & R_1 & R_2 & C & & & & \\
| & | & | & | & | & |{-T} & & & & & & \\
Y_2 & BC & L_1 & R_1^{k-1} & R_2 & C & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & &
\end{array}
$$

Depending on k the bottom section of the tableau fits either case α or case β.

$$
\begin{array}{lll}
k \text{ odd }: & \text{case } \beta. & b(L_2, \sigma) = X_2. \\
k \text{ even }: & \text{case } \alpha. & b(L_2, \sigma) = X_1.
\end{array}
$$

The formula for s is

$$s(L_2) = 1 + \left\lfloor \frac{k}{2} \right\rfloor .$$

Next we find $b(L_1, \sigma)$.

$$
\begin{array}{c|lllllllllllll}
 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_2 & R_3 & L_2 & R_3 & L_2 & R_3 & L_2 \ldots \\
 & | & | & | & | & | & | & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_2 & R_3 & L_2 & R_3 & L_2 & BC & L_1 \ldots \\
| & | & | & | & | & | & | & | & | & | & |{-T} & & \\
X_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_2 & BC & L_1 & R_2 & C & & \\
| & | & | & | & | & |{-T} & & & & & & & \\
Y_2 & BC & L_1 & R_1^{k-2} & R_2 & C & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & &
\end{array}
$$

$$
k \text{ odd }: \quad \text{case } \alpha. \quad b(L_1, \sigma) = X_1.
$$

$$k \text{ even}: \qquad \text{case } \beta. \qquad b(L_1,\sigma) = X_2.$$

The formula for s is

$$s(L_1) = \left\lfloor \frac{k+1}{2} \right\rfloor .$$

Next we find $b(R_2,\sigma)$.

$$
\begin{array}{c|cccccccccccc}
X_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_1 & R_1 & R_2 & BC & L_1 & R_1 & R_2 \\
| & | & | & | & | & | & | & | & | & | & | & |-T & \\
Y_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_2 & R_3 & L_2 & R_3 & L_2 & C & \\
| & | & | & | & | & | & | & | & | & | & |-T & & \\
Y_2 & BC & L_1 & R_1^{k-2} & R_1 & R_1 & R_2 & BC & L_1 & R_2 & C & & \\
| & | & | & | & | & |-T & & & & & & \\
X_2 & BC & L_1 & R_1^{k-2} & R_2 & C & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & &
\end{array}
$$

$$k \text{ odd}: \qquad \text{case } \alpha. \qquad b(R_2,\sigma) = X_1.$$

$$k \text{ even}: \qquad \text{case } \beta. \qquad b(R_2,\sigma) = X_2.$$

Hence, the formula for s is

$$s(R_2) = 1 + \left\lfloor \frac{k+1}{2} \right\rfloor .$$

It remains to find $b(R_1,\sigma)$.

$$
\begin{array}{c|ccccccccccc}
 & {\color{gray}BC} & {\color{gray}L_1} & {\color{gray}R_1^{k-1}} & {\color{gray}R_1} & {\color{gray}R_1} & {\color{gray}R_1} & {\color{gray}R_2} & {\color{gray}BC} & {\color{gray}L_1} & {\color{gray}R_1} & {\color{gray}R_2 \ldots} \\
 & {\color{gray}|} & {\color{gray}|} & {\color{gray}|} & {\color{gray}|} & {\color{gray}|} & {\color{gray}|} & {\color{gray}|} & {\color{gray}|} & {\color{gray}|} & {\color{gray}|} & {\color{gray}|} \\
X_2 & BC & L_1 & R_1^{k-1} & R_1 & R_1 & R_1 & R_2 & BC & L_1 & R_1 & R_1 \ldots \\
| & | & | & | & | & | & | & | & | & | & | & |-T \\
X_2 & BC & L_1 & R_1^{k-1} & R_1 & R_2 & R_3 & L_2 & BC & L_1 & R_2 & C \\
| & | & | & | & | & | & | & |-T & & & \\
X_2 & BC & L_1 & R_1^{k-1} & R_1 & R_1 & R_2 & C & & & & \\
| & | & | & | & | & |-T & & & & & \\
Y_2 & BC & L_1 & R_1^{k-1} & R_2 & C & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & & & &
\end{array}
$$

$$k \text{ odd}: \qquad \text{case } \beta. \qquad b(R_1,\sigma) = X_2.$$

$$k \text{ even}: \qquad \text{case } \alpha. \qquad b(R_1,\sigma) = X_1.$$

Note that the first

$$
\begin{array}{l}
R_1 \\
|-T \\
C
\end{array}
$$

connection does not result in a colour change when is pulled back to the major column. This is because the $R_1 - C$ connection does not bound the major leaf, which it would need to to pull back long in the following major column. Hence, we have

$$s(R_1) = 1 + \left\lfloor \frac{k}{2} \right\rfloor.$$

To recap:

	k odd	k even
$b(L_2, \sigma)$	X_2	X_1
$b(L_1, \sigma)$	X_1	X_2
$b(R_2, \sigma)$	X_1	X_2
$b(R_1, \sigma)$	X_2	X_1

where

$$X = \begin{cases} L & \text{for } \sigma \text{ even} \\ R & \text{for } \sigma \text{ odd.} \end{cases}$$

Similarly for s.

$$\begin{aligned} s(L_2) = s(R_1) &= 1 + \left\lfloor \frac{k}{2} \right\rfloor \\ s(L_1) &= \left\lfloor \frac{k+1}{2} \right\rfloor \\ s(R_2) &= 1 + \left\lfloor \frac{k+1}{2} \right\rfloor \end{aligned}$$

The arguments of those on page 68, to show that any R_2 or L_2 in a major column will pull back as one endpoint of the minor leaf for at least $n-1$ pull backs, mostly hold for this example also. The only difference is that this example places more than one forward image of the minor leaf in what would be the central gap of $L_{1/7}$ (between T and –T).

The challenge is to show that if there is a known row, which matches that of the minor leaf, and a connection, representing a leaf ℓ, in a major column which connects to an element R_2 or L_2 in the row below then the lower row must also pull back as one endpoint of the minor leaf for at least $n-1$ pull-backs.

The leaf ℓ will either be a long leaf, with one endpoint close to each of those of the major leaf, or a short leaf, with both endpoints near a single endpoint of the major leaf. If the leaf is long then, regardless of whether it is close to the periodic or non-periodic major leaf it will pull back to bound μ_p

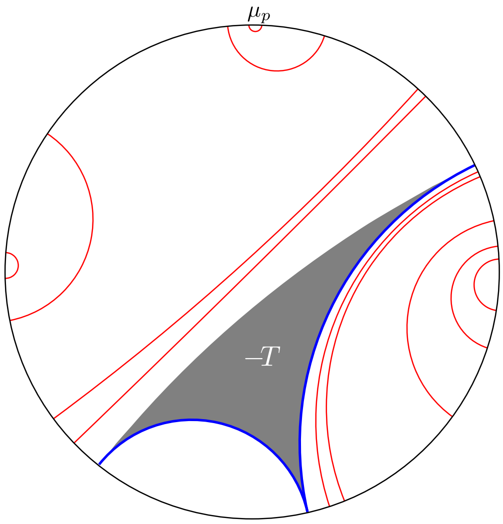

Figure 5.2: The orbit of μ_p, $p = 499/2047$, an example of this family when $k = 1$, in red. The two blue leaves of $-T$ are those corresponding to the $L_2 - C$ and $R_1 - C$ connections.

after three pull backs, and so pull back long in the mid-word colour change column, as it must for both endpoints of the pull-back of ℓ to stay close to the different endpoints of the periodic pre-image of μ_p. If ℓ is short then it bounds no leaf in the forward image of μ_p and so will never pull back long; after each pull-back both endpoints of the pre-image of ℓ will be close to the same endpoint of the periodic pre-image of μ_p.

Hence, the endpoint corresponding to the lower row will pull back close to the pre-image of one endpoint of μ_p for at least $n-1$ pull-backs. As the tableaux above show that all R_2 and L_2 in the major column lie below only other R_2 and L_2 elements this gives that all rows with R_2 or L_2 in a major column pull back as one endpoint of the minor leaf for at least $n-1$ pull-backs. The argument is similar to that on page 68 discussing the same property for the earlier family of examples.

Lemma 5.1.4 *For the examples of $BCR_1^kR_2R_3L_2R_3L_2R_3L_2C$, $k \geq 0$ the count of colour changes in the n^{th} major leaf column, $t(n)$, satisfies*

$$t(n) = \left\lfloor \frac{t(n-1)}{2} \right\rfloor + s(b(n-1)).$$

Proof: The proof of lemma 5.1.4 is similar to that of lemma 5.1.2. However, it is worth noting which rows will pull back long the first time they pull-back into a major column. If the right most connection, representing a leaf ℓ, between two rows is denoted by

$$\begin{array}{c} R_1 \\ | \\ C \end{array}$$

and this connection is in a column containing the $3^{\text{rd}} - (4+k)^{\text{th}}$ forward image of the minor leaf or the column preceding the major column then the first time ℓ pulls back into a major column, if pulled back according to the tableau, it will pull back long.

This is because, similar to the situation in lemma 5.1.2, a leaf ℓ represented by $R_1 - C$ in each of these columns bounds the forward image of μ_p which has an endpoint in the minor row of the same column. Hence, pulling back $R_1 - C$ to the minor column results in a leaf bounding the minor leaf which will then pull back long.

Again, $L_2 - C$ bounds no forward image of the minor leaf and so will never pull back long. ■

It is again possible to combine this information to more precisely state $t(n)$.

$$t(n) = \begin{cases} \frac{t(n-1)-1}{2} + \frac{k-1}{2} + 1 = \frac{t(n-1)+k}{2} & t(n-1), k \text{ odd }, b_{n-1} = R_1 \\ \frac{t(n-1)-1}{2} + \frac{k+1}{2} + 1 = \frac{t(n-1)+k+2}{2} & t(n-1), k \text{ odd }, b_{n-1} = R_2 \\ \frac{t(n-1)}{2} + \frac{k+1}{2} = \frac{t(n-1)+k+1}{2} & t(n-1) \text{ even}, k \text{ odd }, b_{n-1} = L_1 \\ \frac{t(n-1)}{2} + \frac{k-1}{2} + 1 = \frac{t(n-1)+k+1}{2} & t(n-1) \text{ even}, k \text{ odd }, bn - 1 = L_2 \\ \frac{t(n-1)-1}{2} + \frac{k}{2} + 1 = \frac{t(n-1)+k+1}{2} & t(n-1) \text{ odd }, k \text{ even}, b_{n-1} = R_1 \\ \frac{t(n-1)-1}{2} + \frac{k}{2} + 1 = \frac{t(n-1)+k+1}{2} & t(n-1) \text{ odd }, k \text{ even}, b_{n-1} = R_2 \\ \frac{t(n-1)}{2} + \frac{k}{2} = \frac{t(n-1)+k}{2} & t(n-1), k \text{ even}, b_{n-1} = L_1 \\ \frac{t(n-1)}{2} + \frac{k}{2} + 1 = \frac{t(n-1)+k+2}{2} & t(n-1), k \text{ even}, b_{n-1} = L_2. \end{cases}$$

This shows that $t(n)$ will always satisfy

$$\frac{t(n-1)+k}{2} \leq t(n) \leq \frac{t(n-1)+k+2}{2}.$$

and
$$\frac{k - t(n-1)}{2} \leq t(n) - t(n-1) \leq \frac{k + 2 - t(n-1)}{2}$$
Hence, for $t(n-1) < k+2$ we have that $t(n) < k+2$. Also, this shows that $t(n) > t(n-1)$ whenever $t(n-1) < k$, and that there exists an n such that $t(n) = k$.

Assume k odd. If $t(n) = k$ and $b_n = R_1$ then $t(n+1) = k$, $b_{n+1} = R_2$. Then $t(n+1) = k+1$ and $b_n = L_1$, at which point the functions stabilise. If $t(n) = k$ and $b_n = R_2$ then $t(n+1) = k+1$, $b_{n+1} = L_1$, at which point the tableau stabilises.

Assume k even. If $t(n) = k$ and $b_n = L_1$ then $t(n+1) = k$ and $b_n = L_2$. Then $t(n+2) = k+1$ and R_1, at which point the functions stabilise. If $t(n) = k$ and $b_n = L_2$ then $t(n+1) = k+1$ and $b_n = R_1$, at which point the functions stabilise.

Hence, we see that the tableau of any member of this family stabilises with the b_n being one of either L_1 or R_1.

$$\begin{array}{lll} k \text{ odd} & \Rightarrow \exists N: & b_n = L_1 \ \forall n > N \\ & (\Rightarrow \text{ word} & = L_2R_2R_1(L_2R_1)^{\frac{k-1}{2}}L_2BCL_1L_2R_2BCL_1) \\ \\ k \text{ even} & \Rightarrow \exists N: & b_n = R_1 \ \forall n > N. \\ & (\Rightarrow \text{ word} & = BCL_1(L_2R_1)^{\frac{k}{2}}L_2R_1L_2BCL_1L_2R_2) \end{array}$$

Next we derive N using exactly the same method as for the previous family.

Proof: (of theorem 5.1.3) As with the previous family we know that
$$t(n) \geq \frac{t(n-1) + k}{2},$$
and $t(1) \geq k/2$. This gives that
$$t(n) \geq \frac{(2^n - 1)k}{2^n}$$
from which we may deduce N as in the proof of 5.1.1. ■

This shows that, for any minor leaf in this family, the corresponding tableau will converge before
$$\text{period}(\mu_p).N = \text{period}(\mu_p).(\log_2 k + 2)$$
pull backs have been taken.

5.1.3 A Two-Parameter Family

The two parameter example of $BCL_1R_1^{k_1}R_2R_3L_2BCL_1R_1^{k_2}R_2R_3L_2$, with $k_2 < k_1$, is now considered.

There is an extra layer of complication in this example due to the fact that there are two variables in use (k_1 and k_2). Due to this the functions used to calculate the tableau must be modified slightly.

The functions s^1, b^1 along with s^2, b^2, t^2 are now employed. Those with superscript 1 are used as before but to calculate the number of colour changes, and bottom most element, of the column preceding that which contains the *mid-word* BC in the upper-most row. These functions take, as input, values produced by those with the superscript 2, when applied to the previous major leaf column. In turn, the functions with superscript 2 take the output from those with superscript 1 to calculate the number of colour changes, and bottom most element, of a major column.

Once again it is necessary to find all possible values for the numerous functions involved. The functions b^1 and s^1 will be considered first for the sake of clarity. It is worth noting that the lower most sections of all sample sections of the tableau used to calculate the values of b^1 and b^2 must, again, match with one of the two cases

$$
\begin{array}{c}
\text{case } \alpha \\
\begin{array}{c|llll}
\vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & \ldots \\
| & | & | & |-T & \\
X_2 & R_3 & L_2 & C & \\
| & | & |-T & & \\
X_1 & R_2 & C & & \\
 & |T & & &
\end{array}
\end{array}
\qquad
\begin{array}{c}
\text{case } \beta \\
\begin{array}{c|lllll}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
Y_2 & BC & L_1 & R_1 & R_1 & \ldots \\
| & | & | & | & |-T & \\
X_2 & BC & L_1 & R_2 & C & \\
 & & |T & |T & & \\
 & & & & &
\end{array}
\end{array}
$$

which prescribe the way in which any b^1 or b^2 value can arise.

We begin with the situation in which $b_n^2 = L_2$.

$$\begin{array}{llllllll}
X_2 & BC & L_1 & R_1^{k_2-1} & R_1 & R_2 & R_3 & L_2 \\
| & | & | & | & | & | & | & |-T \\
X_2 & BC & L_1 & R_1^{k_2-1} & R_1 & R_1 & R_2 & C \\
| & | & | & | & | & |-T & & \\
Y_2 & BC & L_1 & R_1^{k_2-1} & R_2 & C & & \\
\vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

Then

if k_2 is even then case α. if k_2 is odd then case β.

$$b^1(L_2, \sigma_1) = X_1. \qquad b^1(L_2, \sigma_1) = X_2.$$

Here,

$$s^1(L_2) = 1 + \left\lfloor \frac{k_2}{2} \right\rfloor.$$

Next the situation in which $b_n^2 = L_1$ is examined.

$$\begin{array}{lllllllll}
 & BC & L_1 & R_1^{k_2-2} & R_1 & R_1 & R_2 & R_3 & L_2 \ldots \\
 & | & | & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k_2-2} & R_1 & R_1 & R_2 & BC & L_1 \ldots \\
| & | & | & | & | & |-T & & & \\
Y_2 & BC & L_1 & R_1^{k_2-2} & R_2 & C & & & \\
\vdots & \vdots & \vdots & \vdots & & & & &
\end{array}$$

Then

if k_2 is even then case β. if k_2 is odd then case α.

$$b^1(L_1, \sigma_1) = X_2. \qquad b^1(L_1, \sigma_1) = X_1.$$

Here,

$$s^1(L_1) = \left\lfloor \frac{k_2 + 1}{2} \right\rfloor.$$

Then, to find b_{n+1}^1 when $b_n^2 = R_2$.

$$\begin{array}{lllllll}
 & BC & L_1 & R_1^{k_2} & R_2 & R_3 & L_2 \ldots \\
 & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k_2} & R_1 & R_1 & R_2 \ldots \\
| & | & | & | & | & |-T & \\
Y_2 & BC & L_1 & R_1^{k_2} & R_2 & C & \\
\vdots & \vdots & \vdots & \vdots & & &
\end{array}$$

Then

if k_2 is even then case β. if k_2 is odd then case α.

$$b^1(R_2, \sigma_1) = X_2. \qquad\qquad b^1(R_2, \sigma_1) = X_1.$$

Here,

$$s^1(R_2) = 1 + \left\lfloor \frac{k_2 + 1}{2} \right\rfloor.$$

Then, to find b^1_{n+1} when $b^2_n = R_1$.

$$\begin{array}{cccccccc}
 & BC & L_1 & R_1^{k_2-1} & R_1 & R_1 & R_1 & R_2 \ldots \\
 & | & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k_2-1} & R_1 & R_1 & R_1 & R_1 \ldots \\
| & | & | & | & | & | & | & |-T \\
X_2 & BC & L_1 & R_1^{k_2-1} & R_1 & R_1 & R_2 & C \\
| & | & | & | & | & |-T & & \\
Y_2 & BC & L_1 & R_1^{k_2-1} & R_2 & C & & \\
\vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

Note that the higher $-T$ connection pulls back to a colour change. This is because $k_1 > k_2$ meaning that the leaf given by

$$\begin{array}{ccccccc}
BC & L_1 & R_1^{k_2-1} & R_1 & R_1 & R_1 & R_1 \\
| & | & | & | & | & | & |-T \\
BC & L_1 & R_1^{k_2-1} & R_1 & R_1 & R_2 & C
\end{array}$$

bounds the $(k_2 + 5)^{\text{th}}$ forward image of the minor leaf.

Then

if k_2 is even then case α. if k_2 is odd then case β.

$$b^1(R_1, \sigma_1) = X_1. \qquad\qquad b^1(R_1, \sigma_1) = X_2.$$

Here,

$$s^1(R_1) = 1 + \left\lfloor \frac{k_2}{2} \right\rfloor.$$

The functions with superscript 2 must also be considered. Once more $b^1_n = L_2$ is the first case to be considered.

$$\begin{array}{c|ccccccc}
X_2 & BC & L_1 & R_1^{k_1-1} & R_1 & R_2 & R_3 & L_2 \\
| & | & | & | & | & | & | & |-T \\
X_2 & BC & L_1 & R_1^{k_1-1} & R_1 & R_1 & R_2 & C \\
| & | & | & | & | & |-T & & \\
Y_2 & BC & L_1 & R_1^{k_1-1} & R_2 & C & & \\
\vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

Then

if k_1 is even then case α. if k_1 is odd then case β.

$$b^2(L_2, \sigma_2) = X_1. \qquad b^2(L_2, \sigma_2) = X_2.$$

Here,

$$s^2(L_2) = 1 + \left\lfloor \frac{k_1}{2} \right\rfloor.$$

Next $b_n^1 = L_1$ is examined.

$$\begin{array}{c|cccccccc}
 & BC & L_1 & R_1^{k_1-2} & R_1 & R_1 & R_2 & R_3 & L_2 \dots \\
 & | & | & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k_1-2} & R_1 & R_1 & R_2 & BC & L_1 \dots \\
| & | & | & | & | & |-T & & & \\
Y_2 & BC & L_1 & R_1^{k_1-2} & R_2 & C & & & \\
\vdots & \vdots & \vdots & \vdots & & & & &
\end{array}$$

Then

if k_1 is even then case β. if k_1 is odd then case α.

$$b^2(L_1, \sigma_2) = X_2. \qquad b^2(L_1, \sigma_2) = X_1.$$

Here,

$$s^2(L_1) = \left\lfloor \frac{k_1+1}{2} \right\rfloor.$$

Next $b^2(R_2, \sigma_2)$ is investigated.

$$\begin{array}{c|cccccc}
 & BC & L_1 & R_1^{k_1} & R_2 & R_3 & L_2 \dots \\
 & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k_1} & R_1 & R_1 & R_2 \dots \\
| & | & | & | & | & |-T & \\
Y_2 & BC & L_1 & R_1^{k_1} & R_2 & C & \\
\vdots & \vdots & \vdots & \vdots & & &
\end{array}$$

Then

if k_1 is even then case β. if k_1 is odd then case α.

$$b^2(R_2, \sigma_2) = X_2. \qquad b^2(R_2, \sigma_2) = X_1.$$

Here,

$$s^2(R_2) = 1 + \left\lfloor \frac{k_1+1}{2} \right\rfloor.$$

Lastly, $b^2(R_1, \sigma_2)$ is determined.

$$
\begin{array}{c|ccccccc}
 & BC & L_1 & R_1^{k_1-1} & R_1 & R_1 & R_1 & R_2\ldots \\
 & | & | & | & | & | & | & | \\
X_2 & BC & L_1 & R_1^{k_1-1} & R_1 & R_1 & R_1 & R_1\ldots \\
| & | & | & | & | & | & | & |-T \\
X_2 & BC & L_1 & R_1^{k_1-1} & R_1 & R_1 & R_2 & C \\
| & | & | & | & | & |-T & & \\
Y_2 & BC & L_1 & R_1^{k_1-1} & R_2 & C & & \\
\vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

Note that the first $-T$ connection does not pull back long as the leaf it represents is bounded by the $L_2 - R_2$ leaf in the minor orbit at this position.

Then

if k_1 is even then case α. if k_1 is odd then case β.

$$b^2(R_1, \sigma_2) = X_1. \qquad b^2(R_1, \sigma_2) = X_2.$$

Here,

$$s^2(R_1) = 1 + \left\lfloor \frac{k_1}{2} \right\rfloor .$$

In this family, the formula for $t^2(n)$ can be written in terms of $t^2(n-1)$ and $s^1(n)$ and $s^2(n)$. To do this we employ an intermediary function $u(n)$, writing

$$t^2(n) = \left\lfloor \frac{t^2(n-1) + u(n)}{2} \right\rfloor$$

where

$$u(n) = \begin{cases} s^1(n) + 2s^2(n) & b^2(n-1) \neq R_1 \text{ and } b^2(n) \neq R_1, \\ s^1(n) + 2s^2(n) - 2 & b_2(n-1) = R_1 \text{ or } b^1(n) = R_1 \text{ but not both}, \\ s^1(n) + 2s^2(n) - 4 & b^2(n-1) = R_1 \text{ and } b^1(n) = R_1. \end{cases}$$

Now

$$s^1(n) = s^1(b^2(n-1), k_2), \quad b^1(n) = b^1(b^2(n-1), k_2),$$

$$s^2(n) = s^2(b^1(n), k_1) = s^2(b^2(n-1), k_2, k_1)$$

by abuse of notation. Here, the dependence of $s^1(n)$ on k_2 is only on the parity (odd or even) of k_2, the dependence of $b^1(n)$ on k_2 is only on the parity of both k_2 and $\lfloor k_2/2 \rfloor$ and the dependence of $s^2(n)$ on k_2 and k_1 is only on the parity of each of k_2, $\lfloor k_2/2 \rfloor$ and k_1. Finally we have

$$b^2(n) = b^2(b^1(n), k_1, t^2(n))$$

by abuse of notation, where the dependence on k_1 and $t^2(n)$ is only on their parities.

Here are tables summarising the calculation. We write

$$N = 2\lfloor k_1/2 \rfloor + \lfloor k_2/2 \rfloor$$

k_1, k_2, $\lfloor k_2/2 \rfloor$ odd

X	$b^1(X,k_2)$	$s^1(X,k_2)$	$s^2(X,k_2,k_1)$	$b^2(X,k_2,k_1,t)$	$u(X,k_1,k_2)$
R_1	L_2	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 1$	L_2 or R_2	$N+2$
R_2	L_1	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+4$
L_1	L_1	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+3$
L_2	L_2	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 1$	L_2 or R_2	$N+3$

k_1 even , k_2, $\lfloor k_2/2 \rfloor$ odd

X	$b^1(X,k_2)$	$s^2(X,k_2)$	$s^1(X,k_2,k_1)$	$b^2(X,k_2,k_1,t)$	$u(X,k_1,k_2)$
R_1	L_2	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+2$
R_2	L_1	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor$	L_2 or R_2	$N+2$
L_1	L_1	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor$	L_2 or R_2	$N+1$
L_2	L_2	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+3$

k_1 , k_2 odd, $\lfloor k_2/2 \rfloor$ even

X	$b^1(X,k_2)$	$s^1(X,k_2)$	$s^2(X,k_2,k_1)$	$b^2(X,k_2,k_1,t)$	$u(X,k_1,k_2)$
R_1	R_2	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 2$	L_1 or R_1	$N+4$
R_2	R_1	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 2$	L_2 or R_2	$N+4$
L_1	R_1	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 2$	L_2 or R_2	$N+3$
L_2	R_2	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 2$	L_1 or R_1	$N+5$

k_1 even , k_2 odd, $\lfloor k_2/2 \rfloor$ even

X	$b^1(X,k_2)$	$s^1(X,k_2)$	$s^2(X,k_2,k_1)$	$b^2(X,k_2,k_1,t)$	$u(X,k_1,k_2)$
R_1	R_2	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+2$
R_2	R_1	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 2$	L_2 or R_2	$N+4$
L_1	R_1	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 2$	L_2 or R_2	$N+3$
L_2	R_2	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+3$

k_1 odd , k_2 even, $\lfloor k_2/2 \rfloor$ odd

X	$b^1(X,k_2)$	$s^1(X,k_2)$	$s^2(X,k_2,k_1)$	$b^2(X,k_2,k_1,t)$	$u(X,k_1,k_2)$
R_1	L_1	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+2$
R_2	R_2	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 2$	L_1 or R_1	$N+5$
L_1	R_2	$\lfloor k_2/2 \rfloor$	$\lfloor k_1/2 \rfloor + 2$	L_1 or R_1	$N+4$
L_2	L_1	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+3$

k_1 even , k_2 even, $\lfloor k_2/2 \rfloor$ odd

X	$b^1(X,k_2)$	$s^1(X,k_2)$	$s^2(X,k_2,k_1)$	$b^2(X,k_2,k_1,t)$	$u(X,k_1,k_2)$
R_1	L_1	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor$	L_1 or R_1	N
R_2	R_2	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+3$
L_1	R_2	$\lfloor k_2/2 \rfloor$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+2$
L_2	L_1	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor$	L_1 or R_1	$N+1$

k_1 odd, k_2 even, $\lfloor k_2/2 \rfloor$ even

X	$b^1(X,k_2)$	$s^1(X,k_2)$	$s^2(X,k_2,k_1)$	$b^2(X,k_2,k_1,t)$	$u(X,k_1,k_2)$
R_1	R_1	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 2$	L_2 or R_2	$N+2$
R_2	L_2	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 1$	L_2 or R_2	$N+3$
L_1	L_2	$\lfloor k_2/2 \rfloor$	$\lfloor k_1/2 \rfloor + 1$	L_2 or R_2	$N+2$
L_2	R_1	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 2$	L_2 or R_2	$N+4$

k_1 even, k_2 even, $\lfloor k_2/2 \rfloor$ even

X	$b^1(X,k_2)$	$s^1(X,k_2)$	$s^2(X,k_2,k_1)$	$b^2(X,k_2,k_1,t)$	$u(X,k_1,k_2)$
R_1	R_1	$\lfloor k_2/2 \rfloor + 2$	$\lfloor k_1/2 \rfloor + 2$	L_1 or R_1	$N+2$
R_2	L_2	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+3$
L_1	L_2	$\lfloor k_2/2 \rfloor$	$\lfloor k_1/2 \rfloor + 1$	L_1 or R_1	$N+2$
L_2	R_1	$\lfloor k_2/2 \rfloor + 1$	$\lfloor k_1/2 \rfloor + 2$	L_1 or R_1	$N+4$

The stablising values of $(t^2(n), b^2(n))$ can then be computed as follows. Note that the parity of N is the parity of $\lfloor k_2/2 \rfloor$ in all cases.

Parities	stabilised value of $(t^2(n), b^2(n))$
k_1, k_2, $\lfloor k_2/2 \rfloor$ odd	$(N+3, L_1)$
k_1 even, k_2, $\lfloor k/2 \rfloor$odd	$(N+2, R_1)$
k_1, k_2 odd, $\lfloor k_2/2 \rfloor$ even	$(N+3, R_1)$
k_1 even, k_2 odd, $\lfloor k_2/2 \rfloor$ even	$(N+2, L_2)$
k_1 odd, k_2 even, $\lfloor k_2/2 \rfloor$ odd	$(N+3, L_1)$
k_1 even, k_2 even, $\lfloor k_2/2 \rfloor$ odd	$(N+1, L_1)$
k_1 odd , k_2 even, $\lfloor k_2/2 \rfloor$ even	$(N+3, R_2)$
k_1 even, k_2 even, $\lfloor k_2/2 \rfloor$ even	$(N+2, L_1)$

5.2 The General Case

We now work towards proving theorem 5.2.1 which gives a bound for the number of steps required for convergence in the general case. We begin with a note describing which minor leaves are considered, followed by some definitions before stating the theorem.

Examining the leaves and gaps of $L_{3/7}$ on the interior of S^1 we see that any which are pulled back through UC or BC are pulled back short. Further, they continue to be pulled back short, decreasing in length predictably. While there exist words which do not contain UC or BC almost every valid word of $L_{3/7}$ does. From this point on this section will concern itself only with tableaux for minor leaves labelled by symbolic words containing at least one occurrence of UC or BC.

Recall from chapter 3 that each length n word of the tableau labels a pull-back of the central gap of $L_{3/7}$ and that any length n postfix of a pair of rows of the tableau labels two gaps in $L_{3/7}$ which are connected by an n^{th} pre-image leaf in the boundary of $-T$ (this follows from the fact that each row begins with the central gap connecting to the row above via a leaf of $-T$).

Due to this structure in the tableau it is possible to consider a column in the tableaux as the triangle edges on the exterior of S^1 represented by the pairs of rows, together with the gaps of $L_{3/7}$ on the interior of S^1 which connect them.

As in chapter 3 we use the following notation for individual elements of the tableau: the element $x_{i,j}$ is the j^{th} entry (from the right) of the i^{th} row of the tableau. Define the number r_i such that for any i we have that $x_{i,j}$ is not defined for $j < r_i$.

It is possible to divide the tableau into blocks of columns of width n (the period of μ_p). The elements in the m^{th} block, which will be referred to as C_m, are $x_{i,j}$ where $r_i \leq j$ and $(m-1)n < j \leq mn$. The following theorem further requires that these blocks are in turn separated into β_m, γ_m and Δ_m.

Define Δ_m to be the elements $x_{i,j} \in C_m$ where $(m-1)n < r_i \leq j$. That is, Δ_m consists of rows whose right most element, r_i, lies in block C_m. Define γ_m to be the elements $x_{i,j} \in C_m$ where $(m-2)n < r_i \leq (m-1)n$. Then γ_m consists of the same rows as Δ_{m-1}. Finally, take β_m to be the complement of Δ_m and γ_m in C_m (ignoring rows which have no elements defined in C_m). See figure 5.3 for a visual glossary. It is worth noting that the Δ_m (and so γ_m) consist of at most n rows, due to the fact that $r_i < r_{i+1}$.

We may now state the main theorem of this section.

Theorem 5.2.1 (main theorem) *Consider the tableau of any μ_p of period*

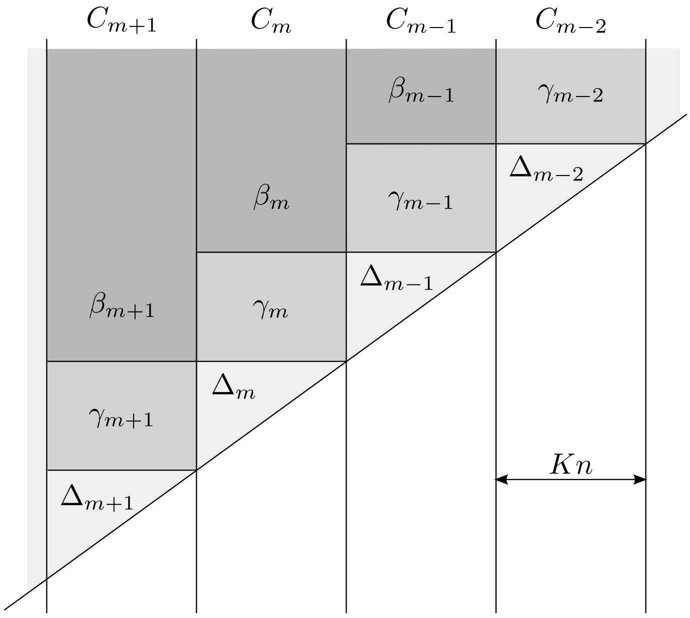

Figure 5.3: Partitioning of tableau

n. The elements of the tableau satisfy

$$x_{i,j} \in \beta_m \Rightarrow x_{i,j} = x_{i,j+ln} \text{ for } l \in \mathbb{N}.$$

A corollary, 5.2.11, of this theorem which uses this result to give a weak bound on the number of columns required in the tableau is presented following the proof of the main theorem.

Before proving theorem 5.2.1 we need numerous other results. We begin by working towards theorem 5.2.10 which states that, for a column in the tableau containing N rows, how many pull backs are required until all rows from this column converge.

Lemma 5.2.2 (centrally enlarging lemma [TH]) *Set L to be an invariant lamination and take $\ell \in L$ with $|\ell| > 1/3$. Then the first forward image of ℓ under the lamination map, s, which lies in the disk, D, bounded by arcs of S^1 together with ℓ and $-\ell$, $s^{\circ j}(\ell)$ say, is such that*

$$|s^j(\ell)| > |\ell|.$$

Proof: For a leaf to lie in D it must have length greater than $|\ell|$ or less than $1/2 - |\ell| < 1 - 2|\ell|$.

Any leaf of length greater than $|\ell|$ must lie in D. Then, all leaves, $s^{\circ i}(\ell)$, $i > 0$, of length less than $|\ell|$ are either the image of a leaf of length $1/3 < |s^{\circ i-1}(\ell)| < |\ell|$, else they are the image of a leaf of length $1-2|\ell| < |s^{\circ i-1}(\ell)| < 1/3$. In both cases they are longer than $1 - 2|\ell|$. ■

Lemma 5.2.3 *Take a pre-periodic infinite sided gap, G, in a lamination with lamination map s. Label the longest leaves in G ℓ_α and ℓ_β. If n is the smallest integer such that $s^{\circ n}(G)$ is the central gap then*

$$s^{\circ i}(\ell_\alpha) \quad \textit{and} \quad s^{\circ i}(\ell_\beta)$$

are the longest leaves of $s^{\circ i}(G)$ for all $0 \le i \le n$.

Proof: Consider ℓ_α only. Some $0 \le i_0 \le n$ will be such that $s^{\circ i_0}(\ell_\alpha)$ will have length greater than $1/3$. If this forward image is not an edge of the central gap there must be an i_1, with $i_0 \le i_1 \le n$, such that $s^{\circ i_1}(\ell_\alpha)$ lies between $s^{\circ i_0}(\ell_\alpha)$ and $-s^{\circ i_0}(\ell_\alpha)$. Continue the sequence of i_j so that $s^{\circ i_j}(\ell_\alpha)$ lies in the region bounded by $s^{\circ i_{j-1}}(\ell_\alpha)$, $-s^{\circ i_{j-1}}(\ell_\alpha)$, and arcs of S^1 until $i_j = n$.

By lemma 5.2.2

$$|s^{\circ i_j}(\ell_\alpha)| < |s^{\circ i_{j+1}}(\ell_\alpha)|$$

giving that

$$\frac{1}{3} < |s^{\circ i_0}(\ell_\alpha)| < |s^{\circ n}(\ell_\alpha)|.$$

As there are only two sides of the central gap of length greater than $1/3$ we see that ℓ_α maps onto one of the two longest leaves of $s^{\circ n}(G)$, the central gap. Similarly for ℓ_β. As $s^{\circ i}|_G$ is homeomorphic for $i \le n$ we see that $s^{\circ n}(\ell_\alpha)$ and $s^{\circ n}(\ell_\beta)$ are the two, distinct, major leaves of the lamination.

If there were any other leaf ℓ of G such that $s^{\circ i}(\ell)$, $i \le n$, were the longest leaf of $s^{\circ i}(G)$ then we could apply the above argument to this leaf to show that $s^{\circ n}(\ell)$ is a third leaf of length greater than $1/3$ on the boundary of the central gap. This is clearly a contradiction, giving the result. ■

The following lemma uses the same mechanisms in the proof as lemma 5.2.3. However, it is concerned with a different case which is encountered in lemma 5.2.6.

Lemma 5.2.4 *Take G to be a periodic gap of period p with three or more sides and let ℓ be any side of G other than its two longest.*

Then, if K is such that

$$s^{\circ K}(\ell)$$

is the first forward image of ℓ equal to the longest leaf on the boundary of G we have that

$$|s^{\circ i-1}(\ell)| < |s^{\circ i}(\ell)|$$

for all $i \leq k$ where $k = K - 2p$.

Proof: Assume ℓ is not strictly increasing as it is mapped onto $s^{\circ i+1}(\ell)$, $i + 1 \leq k$. This implies that $s^{\circ i}(\ell)$ has length greater than 1/3. Label $s^{\circ i}(\ell)$ as ℓ_0. Continue forming this sequence, setting ℓ_{i+1} to be the first image of ℓ_i which lies between ℓ_i and $-\ell_i$. Due to lemma 5.2.2 we have $|\ell_{i+1}| > |\ell_i|$.

As all the $(\ell_i, -\ell_i)$ pairs surround G^*, the gap containing the longest periodic leaf in the forward orbit of G, the first image of ℓ_0 to lie on G^* will be ℓ_n, the final leaf in the sequence $\{\ell_i\}$. This shows that

$$|\ell_n| > |\ell_0| = 1/3.$$

As only two sides of G^* have length greater than 1/3 we see that the first image of ℓ to lie on G^* is either the longest periodic leaf in the orbit or its p^{th} pre-image. This is a contradiction, giving the result. ■

Lemma 5.2.5 *Let y be the length of the third longest leaf in the period p infinite sided central gap, let x be the distance from this leaf to either of the major leaves, and z to be the width of the central gap.*

Then the ratio

$$\frac{x}{y} = \frac{1}{2(2^{p-1} - 1)}.$$

Proof: It is clear that

$$z = y + 2x.$$

Note that under $s^{\circ p}$ the region measured by x is mapped onto that measured by z. Also, lemma 5.2.4 gives that the region measured by x must be mapped homeomorphically by $s^{\circ p}$ so that we have

$$z = 2^p x.$$

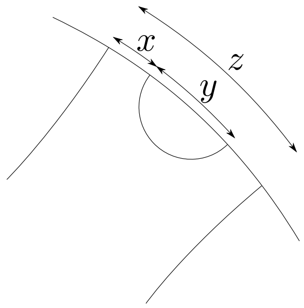

Figure 5.4: The labelling used in lemma 5.2.5. The three leaves drawn are the two major leaves and the next longest edge of the central gap.

Combining these two equations gives

$$\begin{aligned}\frac{x}{y} &= \frac{x}{z-2x} \\ &= \frac{x}{2^p x - 2x} \\ &= \frac{1}{2(2^{p-1}-1)}.\end{aligned}$$

■

Lemma 5.2.6 *Let L_p be a lamination with minor leaf μ_p of period n_p. Let ℓ_1, ℓ_2 be leaves of L_p with endpoints (x_1, y_1) and (x_2, y_2) such that $|\ell_1| > |\ell_2|$. There exists an $\varepsilon(n_p) > 0$ such that if $|x_1 - x_2| \leq \varepsilon Min(|\ell_2|, |y_1 - y_2|)$ then there exists a finite sided gap between ℓ_1 and ℓ_2.*

Proof: Take ℓ_1, ℓ_2 as above. There is a at least one gap between ℓ_1 and ℓ_2 with a vertex between x_1 and x_2 and another between y_1 and y_2. In fact, taking any interval I transversal to ℓ_1, ℓ_2, as exemplified in figure 5.5, the union of the interiors of such gaps has full measure on I (see [TH], chapter 6, part II). We assume that there exists no finite sided gap separating ℓ_1 from ℓ_2. Take $\{G^i\}$ to be the infinite sided gaps bounding ℓ_1 from ℓ_2 which must exist under this assumption. Label the leaf of G^i which has endpoint closest to x_1 to be g_1^i and the leaf of G^i which has endpoint closest to x_2 to be g_2^i (as G^i is infinite sided no two of its sides may share an endpoint so g_1^i, g_2^i are distinct). Take the interval between the endpoint of g_1^i closest to x_1 and the endpoint of g_2^i closest to x_2 to be labelled I_x^i, and similarly the interval

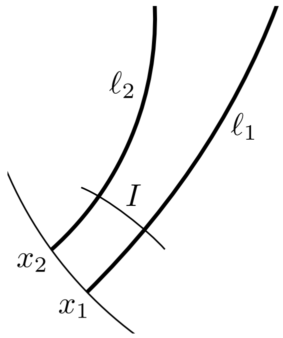

Figure 5.5: A path I crossing between leaves ℓ_1 and ℓ_2.

between the endpoint of g_1^i closest to y_1 and the endpoint of g_2^i closest to y_2 to be labelled I_y^i. Figure 5.6 illustrates this terminology.

As g_2^i is necessarily longer than ℓ_2 and $|I_x^i| < |x_1 - x_2|$ for all i this gives that

$$\begin{aligned} \frac{|I_x^i|}{|g_2^i|} &\leq \frac{|x_1 - x_2|}{|\ell_2|} \\ &\leq \varepsilon. \end{aligned}$$

We now consider the effect of constraining the ratio

$$\frac{|I_x^i|}{|I_y^i|}.$$

The union of the I_x^i must span the interval (x_2, x_1) and that of the corresponding I_y^i must span (y_1, y_2). Therefore, it is not possible for all I_x^i and I_y^i's to satisfy

$$\frac{|I_x^i|}{|I_y^i|} > \frac{|x_1 - x_2|}{|y_2 - y_1|}.$$

Hence, it is also possible to find a $G = G^{i_0}$ with $I_x = I_x^i$, $I_y = I_y^i$ so that

$$\varepsilon \geq \frac{|I_x|}{|I_y|}.$$

We use this G throughout the rest of this proof.

Take $s^{\circ \eta_0}(G) = G'$ be the first forward image of G with a side of length greater than $1/6$. At most two sides may have length greater than $1/6$ as otherwise G', $-G'$ and the pre-images of G' cannot be disjoint.

At this point there are two possible cases to consider:

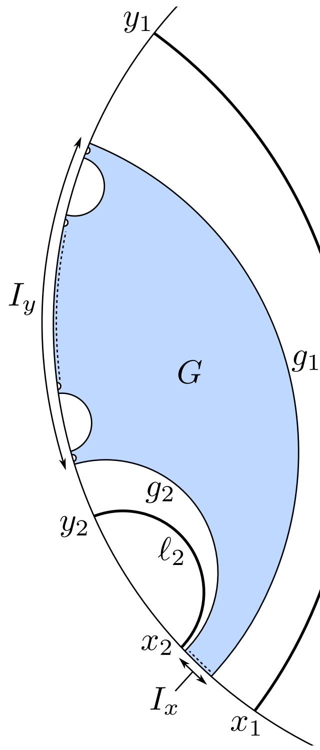

Figure 5.6: An infinite sided gap, G, separating ℓ_1 and ℓ_2 which is assumed to exist.

I) $s^{\circ\eta_0}(g_1)$ and $s^{\circ\eta_0}(g_2)$ are the longest two sides of G'.

II) Otherwise.

In case I the intervals I_x and I_y are considered.

Clearly, in this case, as $G \mapsto G'$ the ratio

$$\frac{|I_x|}{|I_y|} = \frac{|s^{\circ\eta_0}(I_x)|}{|s^{\circ\eta_0}(I_y)|}.$$

This is because none of the sides of the image of G have length greater than $1/6$ before they are mapped to G'. As $s^{\circ j}(g_i)$ has length less than $1/6$ for $i \in \{1, 2\}$, $0 \leq j < \eta_0$, it must also be that $|s^{\circ j}(I_x)|, |s^{\circ j}(I_y)| < 1/6$.

Further, while G' is mapped forward until the image of g_1 is periodic the ratio of the lengths of these intervals will stay constant. This is because if $|I_y|$ decreases (which must happen for the ratio to change) then it previously must have had length greater than $1/3$. The situation where the forward images of g_x, g_y, are the longest sides of this gap and one of the gaps between them has length greater than $1/3$ would result in this forward image of G not being disjoint from its own pre-images, and so cannot occur.

Hence, if $s^{\circ\eta_1}(G)$ contains the first periodic image of g_1 then

$$\frac{|s^{\circ\eta_1}(I_x)|}{|s^{\circ\eta_1}(I_y)|} = \frac{|I_x|}{|I_y|} \leq \varepsilon. \tag{5.2}$$

For any pair of major leaves with endpoints (m_{x1}, m_{y1}) and (m_{x2}, m_{y2}), where $\mathrm{dist}(m_{x1}, m_{x2}) < \mathrm{dist}(m_{x1}, m_{y2})$, the ratio

$$\frac{\mathrm{dist}(m_{x1}, m_{x2})}{\mathrm{dist}(m_{y1}, m_{y2})} = 1. \tag{5.3}$$

We are able to take $\varepsilon < 1$ which forces equation (5.2) to contradict equation (5.3). This shows that case I is not possible.

In case II it is more convenient to examine the ratio

$$\frac{|I_x|}{|g_2|}.$$

For this ratio to change some $s^{\circ i}(g_2)$ must have length greater than $1/3$. Note that only the longest two leaves of a gap may have length greater than $1/3$. Now, g_2 is not one of the two longest leaves of G so that lemma 5.2.3 gives that its forward image will also not be, until after being mapped onto the central gap.

The first forward image of g_2 mapped onto the central gap, $s^{\circ\eta_2}(g_2)$ say, is not one of the two longest leaves of the infinite sided gap containing it. Hence, lemma 5.2.4 gives that if $s^{\circ\eta_3}(g_2)$ is the first image of g_2 to be mapped onto the third longest leaf of the central gap then $s^{\circ\eta_3}$ acts homeomorphically on g_2 and I_x, preserving the ratio

$$\begin{aligned}\frac{|s^{\circ\eta_3}(I_x)|}{|s^{\circ\eta_3}(g_2)|} &= \frac{|I_x|}{|g_2|} \\ &= \varepsilon.\end{aligned}$$

Taking ε to be less than

$$\frac{1}{2(2^{n_p-1}-1)}$$

we see that lemma 5.2.5 states that the result holds in this case also.

For all non-trivial periods, n_p, we have that

$$\frac{1}{2(2^{n_p-1}-1)} < 1$$

so that the result holds, in both cases for

$$\varepsilon < \frac{1}{2(2^{n_p-1}-1)}.$$

■

Lemma 5.2.7 *Take μ, the minor leaf of the lamination L, to be the side of a finite sided periodic gap, G. Take L' to be the lamination produced by removing all forward and backward images of μ from L.*

Then G lies in the infinite sided minor gap, Δ, of L' and

$$\textit{period}(G) = \textit{period}(\Delta).$$

Proof: Note that this construction immediately gives that $G \subset \Delta$. To see that period(G) = period(Δ) we assume that

$$\text{period}(G) \neq \text{period}(\Delta).$$

This is equivalent to assuming that there exists some $s^{\circ n}(G) = G^* \subset \Delta$ also. The gaps G and G^* may or may not be separated by non-periodic pre-images of themselves.

As neither leaves of pre-images of G nor leaves of forward images of G can be accumulation leaves, as G is a finite sided gap in the forward orbit of a minor leaf, there may only be finitely many leaves separating G from G^*. Therefore, there must exist a finite number of infinite sided pre-images of the central gap separating G and G^*.

Map forward until one of these infinite sided gaps is mapped onto the central gap. Then, applying the lamination map once more, both long leaves bounding the central gap are mapped to the same leaf. Hence, the images of two finite sided gaps separating G and G^* are mapped onto each other and one less infinite sided gap is separating the forward images of G and G^*. Continuing this process successively for every such infinite sided gap we reach a position where the image of G coincides with the image of G^*. This is clearly a contradiction giving that G^* cannot exist. ■

Lemma 5.2.8 *Let L be an invariant lamination, with lamination map s_L and minor leaf $\mu_L > \mu_{1/7}$. Take G_i, with $0 < i \leq n$, to be representatives of the periodic cycles of finite sided gaps with G_i closer to μ_L than any other gap in their orbit, of period p_i and bounding G_{i+1} from 0. Set f_i to be the number of sides of G_i.*

Then

$$p_{i+1} > \sum_{m=0}^{i} (f_m - 2)p_m.$$

Also $s_L^j(G_{i+1})$ is adjacent to $s_L^j(G_i)$ for

$$0 \leq j \leq \sum_{m=0}^{i} (f_m - 2)p_m.$$

Proof: Fixing i, let L_i be the lamination with minor leaf μ_i a side of G_i. Take L'_i to be the lamination resulting from removing all forward and backward images of μ_i from L_i. L_i is a tuning of L'_i.

Let Δ'_i be the (infinite sided) minor gap of L'_i. Now $G_i \subset \Delta'_i$, so that we have period$(\Delta'_i) = p_i$ by lemma 5.2.7.

Further, we have that $s_{L'_i}^{\circ p_i} : \Delta'_i \to \Delta'_i$ with degree two (as the only step through which Δ'_i isn't mapped homeomorphically is when its image contains the critical value, when it is mapped with degree two). The map $s_{L'_i}^{\circ p_i}$ also fixes G_i. Hence, there exists a $\varphi : \Delta_i' \to \mathbb{D}$ such that $s \circ \varphi = \varphi \circ s_{L_i'}^{\circ p_i}$ on Δ_i', where $s : z \mapsto z^2$ on S^1.

Consider that $s_{L_i'}^{\circ p_i}$ fixes G_i, giving that $\varphi(G_i)$ must be fixed by s. A consequence of this fact is that $\varphi(\mu_i)$ must be a minor leaf which is not bounded from 0 in the QML (otherwise the orbit of $\varphi(\mu_i)$ under s would not be connected, giving that the orbit of μ_i under $s_{L_i'}^{\circ p_i}$ would also not be connected). Given that $\varphi(\mu_i)$ is also a side of the finite sided gap $\varphi(G_i)$ it cannot be $\mu_{1/3}$, and so cannot span -1.

The $(p_i - 1)^{th}$ image of G_i, G_i^* say, under $s_{L_i'}$ lies in the major gap of L_i'. It is possible to define

$$\varphi^* = \varphi \circ s_{L_i'}^{\circ -(p_i - 1)}$$

so that φ^* acts on the G_i^* in the same way that φ acts on G_i. Then $\varphi^*(G_i^*)$ has only two long sides (that is, longer than 1/3).

Considering the period of μ_i, if this is equal to p_i or $2p_i$ then $f_i = 2$. This gives that G_i is in fact a leaf and so the $(p_i - 1)^{th}$ forward image of μ_i under $s_{L_i'}$ must be the periodic major leaf, m_i.

In all other cases $s_{L_i'}^{\circ p_i - 1}(\mu_i)$ cannot be either of the two long sides of G_i^*.

Lemma 5.2.4 illustrates that if a side of G_i maps forward to a shorter leaf (other than when mapping onto μ_i) it must, on its next visit to G_i^*, map onto one of the two longest leaves in G's orbit. So, all leaves of G_i which don't map onto m_i or the second longest side of G_i^* grow in length through $s_{L_i'}^{j}(G_i)$ for $0 < j \leq (p_i - 1)$. This gives that $s_{L_i'}^{\circ p_i - 1}(\mu_i)$ is the shortest side of G_i^* as μ_i is the shortest side of the finite sided gap G_i.

The leaf $\varphi^*(\, s_{L_i'}^{\circ p_i - 1}(\mu_i)\,)$ is then the shortest side of $\varphi^*(G_i^*)$. Taking a leaf of $\varphi^*(G_i^*)$, if it is a short leaf (of length less than 1/3) then the corresponding leaf of G_i^* can map forward p_i times under $s_{L_i'}$ without danger of becoming long. We may map $\varphi^*(\, s_{L_i'}^{\circ p_i - 1}(\mu_i)\,)$ forward $f_i - 3$ times without it becoming long. This translates into $s_{L_i'}^{\circ p_i - 1}(\mu_i)$ being mapped forward $(f_i - 3)p_i$ times without becoming long. This is equivalent to μ_i being mapped forward $(f_i - 2)p_i - 1$ times under $s_{L_i'}$, which results in an image in G_i^*. As no leaves of G_i^* map long under $s_{L_i'}$ we may map it forward once more, for a total of $(f_i - 2)p_i$ times, without the possibility of the image of μ_i becoming long.

As μ_i is the closest side of G_i to G_{i+1} it is true that the gaps G_i and G_{i+1} can only diverge if $s_{L_i'}^{\circ j}(\mu_i)$ becomes long. Before G_{i+1} completes its orbit it

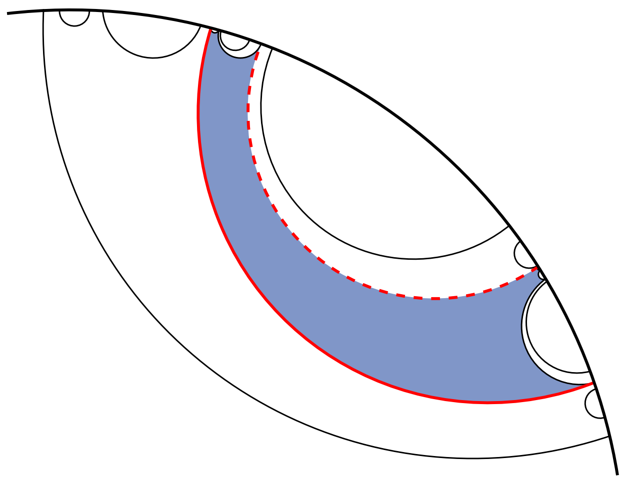

Figure 5.7: Minor gap Δ_i' with the gap G_i superimposed.

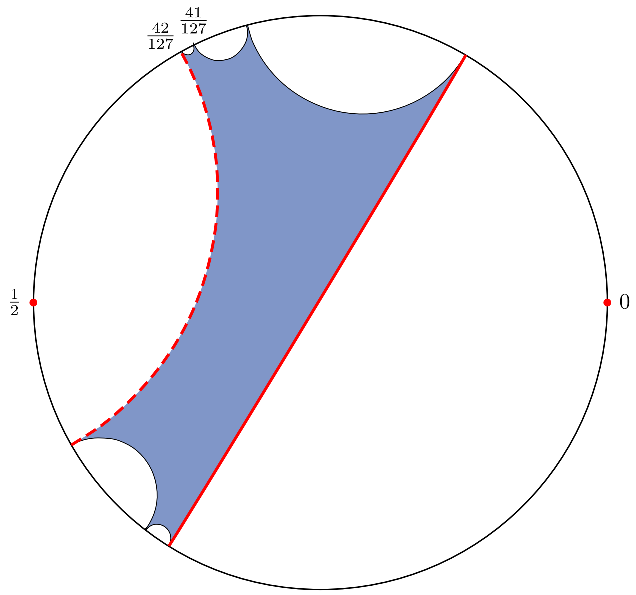

Figure 5.8: The image $\varphi(\Delta_i')$ showing the location of $\varphi(G_i)$. The image, under the map φ, of the individual leaves on the boundary of Δ_i' can be calculated by considering their pre-periodicity. As μ_i' is fixed under $s_{L_i'}^{\circ p_i}$ it is true that $\varphi(\mu_i')$ must also be fixed under s. Hence, $\varphi(\mu_i') = 0$. Similarly, the pre-image of μ_i' (under $s_{L_i'}^{\circ p_i}$) is mapped to $1/2$ by φ, the 2^{nd} pre-images of μ_i' are mapped to $1/4$, and $3/4$, and so on.

must pull away from G_i.

This means that after $(f_i - 2)p_i$ pull backs μ_i may pull back long. However, as this pull back of μ_i lies on a side of G_i it is then possible to take the leaf of G_{i-1} closest to G_i, μ_{i-1}, and apply the same argument. This gives that μ_{i-1} doesn't pull back long for

$$(f_{i-1} - 2)p_{i-1}$$

pull backs.

As the leaf μ_i cannot pull back long unless μ_{i-1} also pulls back long we see that μ_i does not, in fact, pull back long for

$$(f_i - 2)p_i + (f_{i-1} - 2)p_{i-1}$$

pull backs.

Applying this argument iteratively to all of the G_i we see that the result holds. ■

Once again consider a column of the tableau as a collection of leaves in L_p and $L_{3/7}$, with the leaves of $L_{3/7}$ shrinking predictably. Take all of the periodic gaps which the column is following to be $\{G_i\}$, with $\text{period}(G_i) = p_i$ and $p_i > p_j$, $\forall i > j$. Set $p_{\max}$ be the highest period; that of $G_{\max}$.

Theorem 5.2.9 *Given a column which is following the periodic gaps $\{G_i\}$, as above, pulling back $2p_{max}$ times will result in the column having stopped following either G_{max} or G_i, $\forall i \neq max$.*

Proof: Pulling the column back $p_{\max}$ times will mean that, unless the column has stopped following $G_{\max}$, $G_{\max}$ will have, at some point, been minor. At this point all other G_i will have been to one side of $G_{\max}$.

Furthermore, all other gaps of the same period of $G_{\max}$ will have been minor or ceased being followed by the column, meaning that all G_i will be to one side of them also. This gives that after $p_{\max}$ pull backs all $G_i \neq G_{\max}$ will be to one side of $G_{\max}$ and that no other gaps of the same period as $G_{\max}$ separate any of the G_i.

Once this state is achieved a further p_{max} pullbacks will force the column to follow either G_{max} or any of the G_i, $i < \max$ (as a gap of lower period may not follow G_{max} for p_{max} steps). ■

Given a maximum period for the G_i (dictated by the period of μ_p) it is then possible to define an upper bound of the length of time a given column may follow the gaps $\{G_i\}$.

If the period of $\mu_p = n$ then this is the highest period of any leaf in L_p. As the G_i must have at least three sides the period of G_i, p_i say, must be less than, or equal to,

$$\left\lfloor \frac{n}{3} \right\rfloor .$$

Further, lemma 5.2.8 gives that

$$|p_i - p_{i-1}| = (f_{i-1} - 2)p_{i-1}$$

which, as $f_i \geq 3$, implies that

$$\begin{aligned} p_i - p_{i-1} &\geq p_{i-1} \\ p_i &\geq 2p_{i-1}. \end{aligned}$$

Hence, lemma 5.2.8 gives that if $G_{\max}$, with period $p_{\max}$, is the highest period gap following a column then the gap of next highest period will have period

$$p_{\max-1} \leq \frac{p_{\max}}{2}.$$

Continuing this way we see that there are

$$J = \left\lfloor \log_2 \left(\frac{n}{3} \right) \right\rfloor$$

gaps of lower period, at most, also being followed by the column.

An upper bound, then, on the number of steps it takes for a column of a tableau on a lamination with minor leaf of period n to pull back long is

$$B(n) = 2 \sum_{j=0}^{J} \left\lfloor \frac{n}{3 \times 2^j} \right\rfloor .$$

Further, we see that

$$\begin{aligned} B(n) &\leq 2 \sum_{j=0}^{J} \frac{n}{3 \times 2^j} \\ &= \frac{2n}{3} s \end{aligned}$$

where

$$\begin{aligned} s &= \sum_{j=0}^{J} \frac{1}{2^j} \\ &= 2 - \frac{1}{2^J}. \end{aligned}$$

However,

$$\begin{aligned} \frac{1}{2^J} &= \frac{1}{\left\lfloor \frac{n}{3} \right\rfloor} \\ &\geq \frac{3}{n}. \end{aligned}$$

Therefore,

$$\begin{aligned} B(n) &\leq \frac{2n}{3}\left(2 - \frac{1}{2^J}\right) \\ &\leq \frac{2n}{3}\left(2 - \frac{3}{n}\right) \\ &= \frac{4n}{3} - 2. \end{aligned} \tag{5.4}$$

From the above work we have a bound for the number of pull backs which are required for leaves from set rows of a given column to be pulled back to leaves adjacent to μ_p. These rows of the tableau, at this point, may still not be periodic as they may currently be approximating one endpoint of μ_p but pull back so that they approximate the other.

Theorem 5.2.10 *The number of pull backs required so that the block of N rows which are not separated from the orbit of μ by any periodic finite sided gaps become periodic over all further pull backs is*

$$n\lceil \log_2(N) \rceil.$$

Proof: To clearly explain this, a new notation will be used. Previously the letters in the tableau have represented the regions on S^1, and the leaves connecting those regions were implicit in the lettering of two adjacent rows. Now, however, it is useful to consider the leaves between the rows, which we will label with a new lettering.

Note that we are concerned only with the major leaf columns at this point. The properties of long leaves in these columns are harnessed to prove theorem 5.2.10. Note that in all of the following short leaves (that is, leaves connecting L_2 to L_2 or R_2 to R_2) in the major columns are ignored.

The following lettering will be used

$$S - \text{ a long stable leaf,}$$
$$U - \text{ a long unstable leaf.}$$

Here, a *long stable leaf* means a leaf connecting L_2 to R_2 which, after pulling back n times, staying close to the periodic minor orbit at each step, pulls back to a leaf also connecting L_2 to R_2.

A *long unstable leaf* means a leaf connection L_2 to R_2 which, after pulling back n times, staying close to the periodic minor orbit at each step, pulls back to a leaf which does not connect L_2 to R_2.

Next, sets of adjacent long leaves with the same labelling, S or U, are grouped into equivalence classes, $[S]$ or $[U]$ respectively. This is useful as whether or not a leaf is stable or unstable depends on the parity of the count of long leaves above it in a column. Hence, if one leaf switches from stable to unstable, so must any adjacent (ignoring short leaves) similarly labelled long leaves.

As an illustration:

$$
\begin{array}{rc|c|rc|c|rc}
\ldots & R_2 & & & & & & \\
 & | & & \ldots & U & & \ldots & [U] \\
\ldots & L_2 & & & & & & \\
 & | & & & & & & \\
\ldots & L_2 & & & & & & \\
 & | & & \ldots & S & & & \\
\ldots & R_2 & & & & & & \\
 & | & & \ldots & S & & \ldots & [S] \\
\ldots & L_2 & & & & & & \\
 & | & & \ldots & S & & & \\
\ldots & R_2 & & & & & & \\
 & | & & & & & & \\
\ldots & R_2 & & & & & & \\
 & | & & \ldots & U & & \ldots & [U] \\
 & \vdots & \text{becomes} & & \vdots & \text{becomes} & & \vdots \\
 & | & & \ldots & U & & & \\
\ldots & L_2 & & & & & \ldots & [U] \\
 & | & & \ldots & U & & & \\
\ldots & R_2 & & & & & & \\
 & | & & & & & & \\
\ldots & R_2 & & & & & & \\
 & | & & \ldots & S & & \ldots & [S] \\
\ldots & L_2 & & & & & & \\
 & | & & \ldots & U & & \ldots & [U] \\
\ldots & R_2 & & & & & & \\
 & | & & \ldots & S & & \ldots & [S] \\
\ldots & L_2 & & & & & & \\
 & | & & & & & & \\
\ldots & L_2 & & & & & &
\end{array}
$$

Beginning with a column such as that described above, with N' rows, each n pull-backs will reduce the number of equivalence classes by at least

$$\left\lfloor \frac{N'}{2} \right\rfloor .$$

Hence, beginning with N rows gives a maximum of N equivalence classes. Therefore, after

$$n\lceil \log_2(N) \rceil$$

pull backs there will only be one equivalence class remaining, which must necessarily be a class of stable leaves. ■

To conclude this section we combine various results from above to describe certain aspects of the tableaux. The most complete description of the tableaux is given in theorem 5.2.1 which may now be proved.

Proof: (of main theorem) Consider the $N \leq n$ rows of a γ_m. Assume that each γ_m lies immediately below a row which has the symbolic labelling of one endpoint of μ_p. Equation 5.4 shows that it takes less than, or equal to,

$$\frac{4}{3}n - 2$$

pull backs for all rows in a block such as γ_m to pull back to leaves approximating the minor leaf. The next step is to incorporate the number of pull backs necessary for each of these leaves to approximate one endpoint of the minor leaf. Theorem 5.2.10 gives that the required number is

$$n\lceil \log_2 N \rceil.$$

By summing the required pull backs discussed above we get

$$\text{Required pull backs } \leq \frac{4}{3}n - 2 + n\lceil \log_2 N \rceil.$$

Note that $N \leq n$. Also, as we wish that the required number of pull backs are provided by pulling back a row through the n columns of a γ_m this gives that we require

$$n \leq \frac{4}{3}n - 2 + n\lceil \log_2(n) \rceil,$$

which holds for all non-trivial values of n.

If the assumption that γ_m lies immediately below a row which has the symbolic labelling of one endpoint of μ_p holds for $m = m_0$ then the working above shows that β_{m_0+1} consists of rows of symbolic lettering matching that of one endpoint of the minor leaf. It follows that the assumption holds for $m = m_0 + 1$. As the top row of the tableau labels one endpoint of the minor leaf we have that the assumption holds for $m = 2$, so we are done. ■

Corollary 5.2.11 gives uses the main theorem to give a weak bound on the number of columns necessary in a tableau.

Corollary 5.2.11 *A tableau needs at most*

$$\mathcal{O}(n2^{2n})$$

columns, where n is the period of μ_p, to give μ_q.

Proof: Theorem 5.2.1 gives that any row, i_0, in the tableau is periodic by column $(r_{i_0}+2n)$. Hence, a row is fully determined by its first $2n+1$ elements. As there are two possible choices of symbolic letter when pulling back another there are a possible 2^{2n+1} distinct rows. This gives that once a tableau has $2^{2n+1}+1$ rows some row must have occurred twice. Lemma 3.1.1 (stating that the tableau becomes periodic with period dividing n after the first occurrence of repeating rows) then gives that the tableau is periodic from column $r_{(2^{2n+1})}$ at the latest.

As $\mu_p \neq \mu_{1/7}$ we see that any word which is following one endpoint of μ_p can't consist solely of the labels BC, L_1, and R_2. Hence, a row which is following an endpoint of μ_p must connect down to a new row within n columns. Equation 5.4, page 102, then gives that

$$r_{i+1} \leq r_i + \frac{4n}{3} - 2 + n.$$

Using this equation we see that

$$r_{(2^{2n+1})} \leq 2^{2n+1}\left(\frac{4n}{3} - 2 + n\right)$$

giving the result. ∎

Appendix A

Computer Programs

There are a number of computer programs which have been written to complement this thesis. A brief description of each is presented here to assist an interested party in finding anything which is reusable. It is the intention of the author to supply the relevant source code to the University of Liverpool research archive where it will be freely available.

The programs are written in the `C++` programming language. All programs require the `GNUMP` library, used for multiple precision arithmetic, which is freely available for *most* platforms. For all programs which produce graphical output a modified version of the `LibBoard` library is required. This will be supplied with the code in the University of Liverpool research archive. The code was written and compiled on linux using `g++` but, as far as I know, should compile on other platforms using other compilers, provided the necessary libraries are installed. All programs are written for use on the command line.

The principal program presented here is `algorithm.bin`, which is an implementation of the algorithm described in chapter 3. Many other programs were produced as tools to assist with aspects of this thesis, such as generating the laminations or to assist with producing graphics.

While every effort has been made to ensure the programs work consistently it may well be the case that the output is incorrect for sufficiently large input (very high period μ_p, for example), despite using `GNUMP`, due to some fault of the author. Beware.

All rational numbers should be entered as `x/y`, where `x` and `y` are decimal numbers, and with no spaces. For details of installation and prerequisites see the file `README`, supplied with the source code.

`algorithm.bin`

This program asks the user for both endpoints of μ_p, for your chosen p (the program `calc_ml.bin` can be used to find the second endpoint of a minor leaf). The user is then asked "How many full periods of the tableau would you like to calculate?". Three is usually a safe choice but use more if unsure.

The tableau generated by applying the algorithm to this μ_p is printed, along with various other statistics, followed by the word in the symbolic dynamics of $L_{1/7}$ produced by this tableau, followed by the value of the appropriate μ_q.

`calc_ml.bin`

This program uses an algorithm from [TH] to calculate a second endpoint of a minor leaf.

`calc_qml.bin`

This program calculates all periodic minor leaves of a given period. It asks the user for the period and then writes the leaves to a file called `qml_p`n`.txt`, for n the period. This program uses the same algorithm as `calc_ml.bin`, as this is significantly faster than Lavaurs' algorithm.

`draw_qml.bin`

This draws all leaves in the files `qml_p`n`.txt` for $n \in \mathbb{N}$, $2 \leq n \leq N$ where N is the lowest integer such that `qml_p`N`.txt` doesn't exist.

`pre-images.bin` and `pre-images-outside.bin`

After asking for the endpoints of a major leaf and both endpoints of a second leaf, the program prompts the user for how many pre-images of this second leaf should be drawn. The difference between `pre-images.bin` and `pre-images-outside.bin` is that `pre-images.bin` draws the leaves on the interior of S^1 while `pre-images-outside.bin` draws the leaves on the exterior of S^1. The output is written to `lamination.svg` and `lamination-outside.svg`, respectively.

There are a number of lesser programs included also.

The author apologises wholeheartedly to those who dare read the source code.

Appendix B

Equivalent matings

Here, for periods 4–14, the endpoints of μ_p are catalogued allong with the endpoints of the equivalent μ_q.

B.1 Period 4

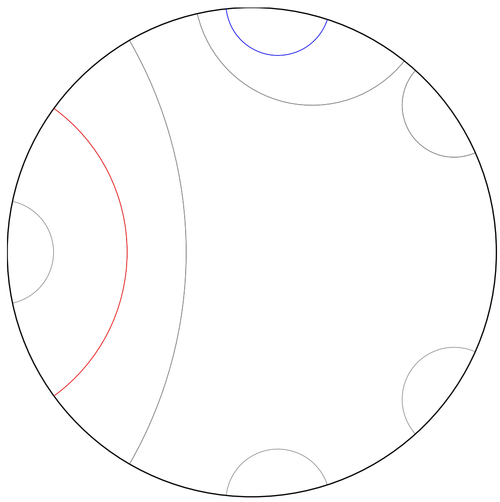

Figure B.1: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 4.

$$\frac{3}{15}, \frac{4}{15} \quad \simeq \quad \frac{3}{5}, \frac{2}{5}$$

B.2 Period 5

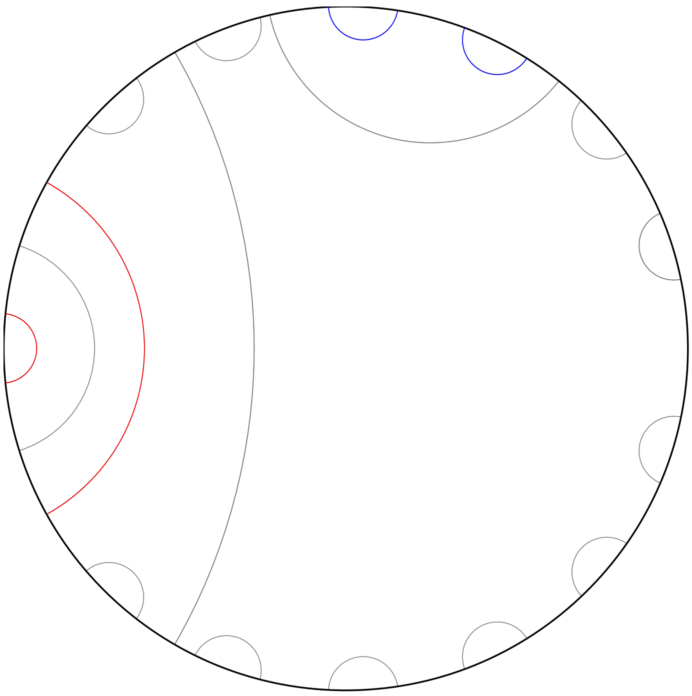

Figure B.2: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 5.

$$\frac{5}{31}, \frac{6}{31} \simeq \frac{15}{31}, \frac{16}{31}$$
$$\frac{7}{31}, \frac{8}{31} \simeq \frac{18}{31}, \frac{13}{31}$$

B.3 Period 6

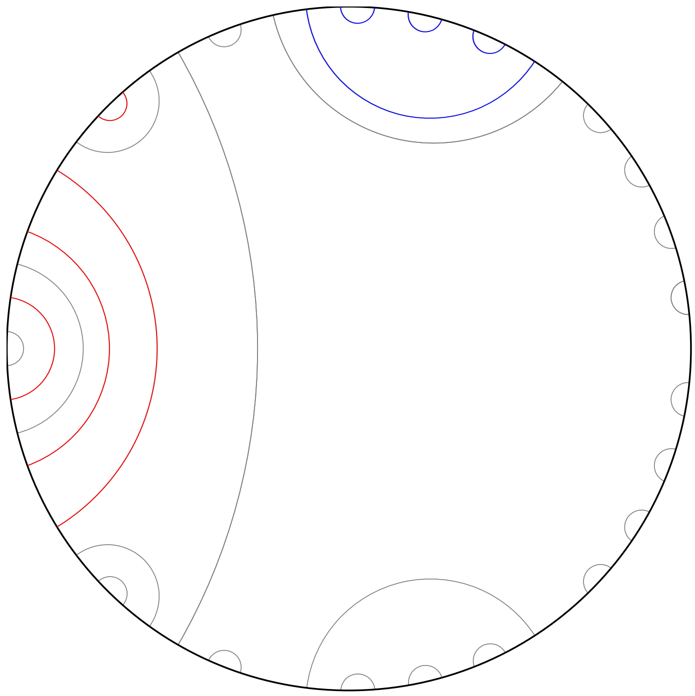

Figure B.3: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 6.

$$
\begin{aligned}
\frac{10}{63}, \frac{17}{63} &\simeq \frac{4}{9}, \frac{5}{9} \\
\frac{11}{63}, \frac{12}{63} &\simeq \frac{10}{21}, \frac{11}{21} \\
\frac{13}{63}, \frac{14}{63} &\simeq \frac{23}{63}, \frac{8}{21} \\
\frac{15}{63}, \frac{16}{63} &\simeq \frac{37}{63}, \frac{26}{63}
\end{aligned}
$$

B.4 Period 7

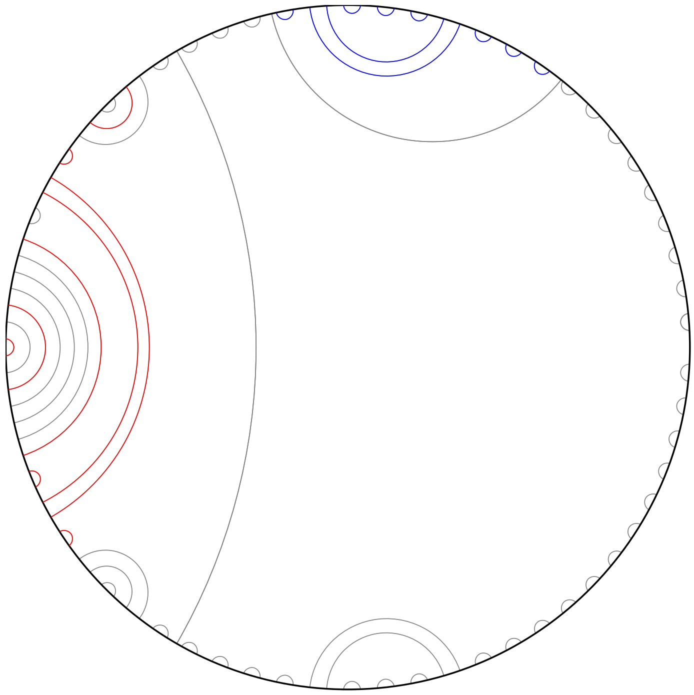

Figure B.4: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 7.

$$\frac{19}{127}, \frac{20}{127} \simeq \frac{57}{127}, \frac{70}{127}$$
$$\frac{21}{127}, \frac{22}{127} \simeq \frac{63}{127}, \frac{64}{127}$$
$$\frac{23}{127}, \frac{24}{127} \simeq \frac{61}{127}, \frac{66}{127}$$
$$\frac{25}{127}, \frac{34}{127} \simeq \frac{75}{127}, \frac{76}{127}$$
$$\frac{26}{127}, \frac{33}{127} \simeq \frac{51}{127}, \frac{52}{127}$$

$$\frac{27}{127}, \frac{28}{127} \simeq \frac{49}{127}, \frac{46}{127}$$
$$\frac{29}{127}, \frac{30}{127} \simeq \frac{71}{127}, \frac{72}{127}$$
$$\frac{31}{127}, \frac{32}{127} \simeq \frac{74}{127}, \frac{53}{127}$$
$$\frac{35}{127}, \frac{36}{127} \simeq \frac{73}{127}, \frac{54}{127}$$

B.5 Period 8

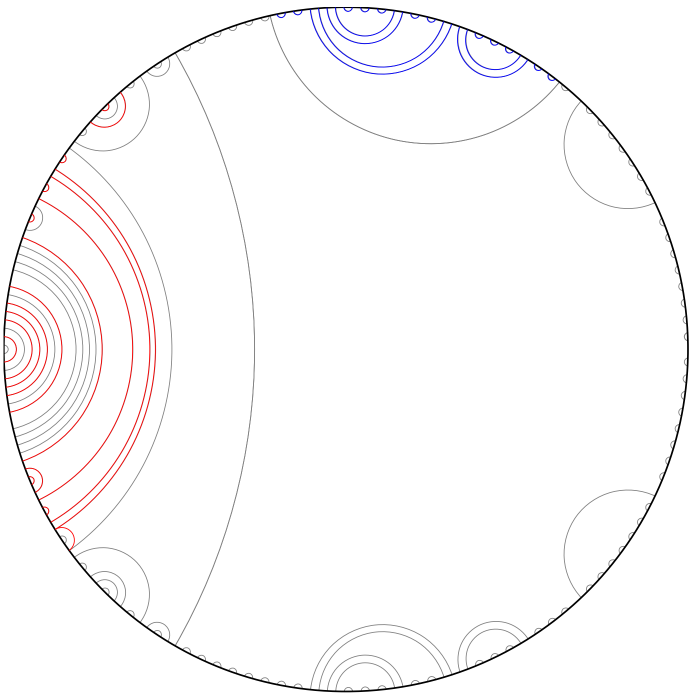

Figure B.5: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 8.

$$\frac{37}{255}, \frac{38}{255} \simeq \frac{37}{85}, \frac{112}{255}$$
$$\frac{39}{255}, \frac{40}{255} \simeq \frac{47}{85}, \frac{38}{85}$$
$$\frac{41}{255}, \frac{50}{255} \simeq \frac{41}{85}, \frac{44}{85}$$
$$\frac{42}{255}, \frac{49}{255} \simeq \frac{124}{255}, \frac{131}{255}$$
$$\frac{43}{255}, \frac{44}{255} \simeq \frac{42}{85}, \frac{43}{85}$$
$$\frac{45}{255}, \frac{46}{255} \simeq \frac{9}{17}, \frac{8}{17}$$
$$\frac{47}{255}, \frac{48}{255} \simeq \frac{133}{255}, \frac{122}{255}$$
$$\frac{51}{255}, \frac{68}{255} \simeq \frac{3}{5}, \frac{10}{17}$$
$$\frac{52}{255}, \frac{67}{255} \simeq \frac{7}{17}, \frac{10}{17}$$
$$\frac{53}{255}, \frac{54}{255} \simeq \frac{19}{51}, \frac{32}{85}$$
$$\frac{55}{255}, \frac{56}{255} \simeq \frac{98}{255}, \frac{31}{85}$$
$$\frac{57}{255}, \frac{66}{255} \simeq \frac{36}{85}, \frac{107}{255}$$
$$\frac{58}{255}, \frac{65}{255} \simeq \frac{49}{85}, \frac{148}{255}$$
$$\frac{59}{255}, \frac{60}{255} \simeq \frac{29}{51}, \frac{142}{255}$$
$$\frac{61}{255}, \frac{62}{255} \simeq \frac{104}{255}, \frac{103}{255}$$
$$\frac{63}{255}, \frac{64}{255} \simeq \frac{149}{255}, \frac{106}{255}$$
$$\frac{69}{255}, \frac{70}{255} \simeq \frac{143}{255}, \frac{48}{85}$$
$$\frac{71}{255}, \frac{72}{255} \simeq \frac{146}{255}, \frac{109}{255}$$

B.6 Period 9

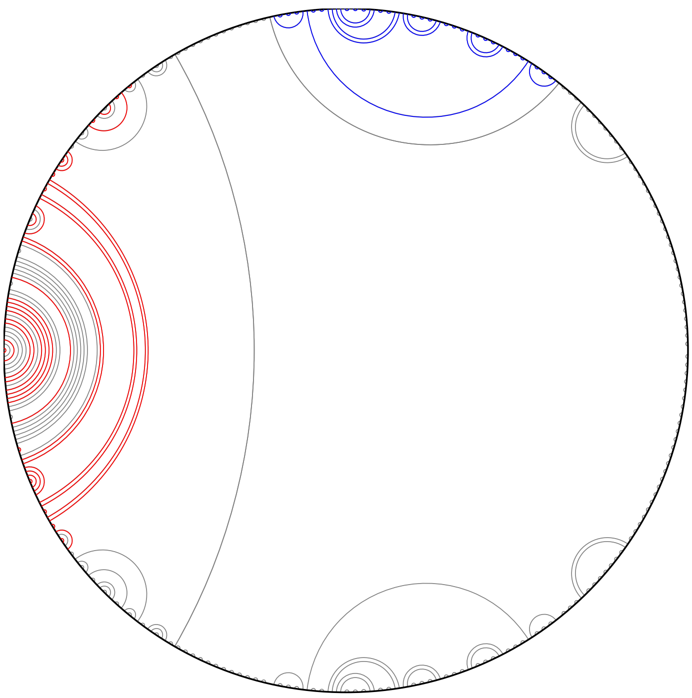

Figure B.6: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 9.

$$\frac{74}{511}, \frac{81}{511} \simeq \frac{220}{511}, \frac{227}{511}$$
$$\frac{75}{511}, \frac{76}{511} \simeq \frac{222}{511}, \frac{225}{511}$$
$$\frac{77}{511}, \frac{78}{511} \simeq \frac{279}{511}, \frac{40}{73}$$
$$\frac{79}{511}, \frac{80}{511} \simeq \frac{282}{511}, \frac{229}{511}$$
$$\frac{82}{511}, \frac{137}{511} \simeq \frac{228}{511}, \frac{283}{511}$$
$$\frac{83}{511}, \frac{84}{511} \simeq \frac{249}{511}, \frac{262}{511}$$
$$\frac{85}{511}, \frac{86}{511} \simeq \frac{255}{511}, \frac{256}{511}$$
$$\frac{87}{511}, \frac{88}{511} \simeq \frac{253}{511}, \frac{258}{511}$$
$$\frac{89}{511}, \frac{98}{511} \simeq \frac{244}{511}, \frac{267}{511}$$
$$\frac{90}{511}, \frac{97}{511} \simeq \frac{243}{511}, \frac{268}{511}$$
$$\frac{91}{511}, \frac{92}{511} \simeq \frac{34}{73}, \frac{39}{73}$$
$$\frac{93}{511}, \frac{94}{511} \simeq \frac{248}{511}, \frac{263}{511}$$
$$\frac{95}{511}, \frac{96}{511} \simeq \frac{38}{73}, \frac{35}{73}$$
$$\frac{99}{511}, \frac{100}{511} \simeq \frac{246}{511}, \frac{265}{511}$$
$$\frac{101}{511}, \frac{102}{511} \simeq \frac{303}{511}, \frac{304}{511}$$
$$\frac{103}{511}, \frac{104}{511} \simeq \frac{30}{73}, \frac{205}{511}$$
$$\frac{105}{511}, \frac{114}{511} \simeq \frac{187}{511}, \frac{188}{511}$$
$$\frac{106}{511}, \frac{113}{511} \simeq \frac{195}{511}, \frac{28}{73}$$

$$\frac{107}{511}, \frac{108}{511} \simeq \frac{190}{511}, \frac{193}{511}$$
$$\frac{109}{511}, \frac{110}{511} \simeq \frac{183}{511}, \frac{184}{511}$$
$$\frac{111}{511}, \frac{112}{511} \simeq \frac{197}{511}, \frac{186}{511}$$
$$\frac{115}{511}, \frac{132}{511} \simeq \frac{42}{73}, \frac{31}{73}$$
$$\frac{116}{511}, \frac{131}{511} \simeq \frac{297}{511}, \frac{214}{511}$$
$$\frac{117}{511}, \frac{118}{511} \simeq \frac{41}{73}, \frac{288}{511}$$
$$\frac{119}{511}, \frac{120}{511} \simeq \frac{290}{511}, \frac{285}{511}$$
$$\frac{121}{511}, \frac{130}{511} \simeq \frac{300}{511}, \frac{299}{511}$$
$$\frac{122}{511}, \frac{129}{511} \simeq \frac{212}{511}, \frac{211}{511}$$
$$\frac{123}{511}, \frac{124}{511} \simeq \frac{209}{511}, \frac{206}{511}$$
$$\frac{125}{511}, \frac{126}{511} \simeq \frac{296}{511}, \frac{295}{511}$$
$$\frac{127}{511}, \frac{128}{511} \simeq \frac{298}{511}, \frac{213}{511}$$
$$\frac{133}{511}, \frac{134}{511} \simeq \frac{207}{511}, \frac{208}{511}$$
$$\frac{135}{511}, \frac{136}{511} \simeq \frac{306}{511}, \frac{43}{73}$$
$$\frac{138}{511}, \frac{145}{511} \simeq \frac{284}{511}, \frac{291}{511}$$
$$\frac{139}{511}, \frac{140}{511} \simeq \frac{286}{511}, \frac{289}{511}$$
$$\frac{141}{511}, \frac{142}{511} \simeq \frac{215}{511}, \frac{216}{511}$$
$$\frac{143}{511}, \frac{144}{511} \simeq \frac{293}{511}, \frac{218}{511}$$

B.7 Period 10

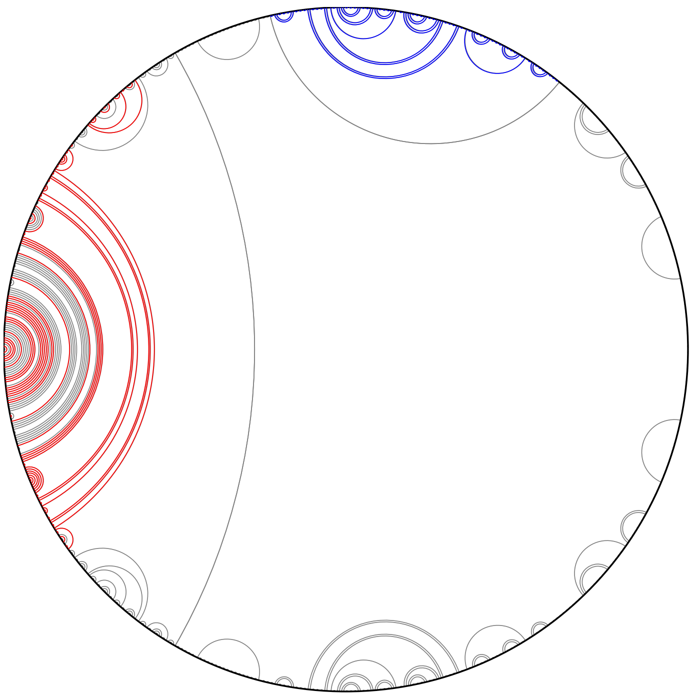

Figure B.7: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 10.

$$\frac{147}{1023},\frac{148}{1023} \simeq \frac{147}{341},\frac{454}{1023}$$
$$\frac{149}{1023},\frac{150}{1023} \simeq \frac{149}{341},\frac{448}{1023}$$
$$\frac{151}{1023},\frac{152}{1023} \simeq \frac{445}{1023},\frac{150}{341}$$
$$\frac{153}{1023},\frac{162}{1023} \simeq \frac{153}{341},\frac{188}{341}$$
$$\frac{154}{1023},\frac{161}{1023} \simeq \frac{460}{1023},\frac{563}{1023}$$
$$\frac{155}{1023},\frac{156}{1023} \simeq \frac{6}{11},\frac{5}{11}$$
$$\frac{157}{1023},\frac{158}{1023} \simeq \frac{455}{1023},\frac{152}{341}$$
$$\frac{159}{1023},\frac{160}{1023} \simeq \frac{565}{1023},\frac{458}{1023}$$
$$\frac{163}{1023},\frac{164}{1023} \simeq \frac{566}{1023},\frac{457}{1023}$$
$$\frac{166}{1023},\frac{197}{1023} \simeq \frac{16}{33},\frac{17}{33}$$
$$\frac{167}{1023},\frac{168}{1023} \simeq \frac{166}{341},\frac{175}{341}$$
$$\frac{169}{1023},\frac{178}{1023} \simeq \frac{169}{341},\frac{172}{341}$$
$$\frac{170}{1023},\frac{177}{1023} \simeq \frac{508}{1023},\frac{515}{1023}$$
$$\frac{171}{1023},\frac{172}{1023} \simeq \frac{170}{341},\frac{171}{341}$$
$$\frac{173}{1023},\frac{174}{1023} \simeq \frac{503}{1023},\frac{520}{1023}$$
$$\frac{175}{1023},\frac{176}{1023} \simeq \frac{46}{93},\frac{47}{93}$$
$$\frac{179}{1023},\frac{180}{1023} \simeq \frac{162}{341},\frac{179}{341}$$
$$\frac{181}{1023},\frac{182}{1023} \simeq \frac{181}{341},\frac{544}{1023}$$

$$\frac{183}{1023},\frac{184}{1023} \simeq \frac{159}{341},\frac{182}{341}$$
$$\frac{185}{1023},\frac{194}{1023} \simeq \frac{491}{1023},\frac{532}{1023}$$
$$\frac{186}{1023},\frac{193}{1023} \simeq \frac{164}{341},\frac{177}{341}$$
$$\frac{187}{1023},\frac{188}{1023} \simeq \frac{497}{1023},\frac{526}{1023}$$
$$\frac{189}{1023},\frac{190}{1023} \simeq \frac{487}{1023},\frac{536}{1023}$$
$$\frac{191}{1023},\frac{192}{1023} \simeq \frac{533}{1023},\frac{490}{1023}$$
$$\frac{195}{1023},\frac{196}{1023} \simeq \frac{178}{341},\frac{163}{341}$$
$$\frac{199}{1023},\frac{200}{1023} \simeq \frac{530}{1023},\frac{493}{1023}$$
$$\frac{201}{1023},\frac{274}{1023} \simeq \frac{201}{341},\frac{604}{1023}$$
$$\frac{202}{1023},\frac{273}{1023} \simeq \frac{611}{1023},\frac{204}{341}$$
$$\frac{203}{1023},\frac{204}{1023} \simeq \frac{202}{341},\frac{203}{341}$$
$$\frac{205}{1023},\frac{206}{1023} \simeq \frac{37}{93},\frac{136}{341}$$
$$\frac{207}{1023},\frac{208}{1023} \simeq \frac{421}{1023},\frac{410}{1023}$$
$$\frac{209}{1023},\frac{266}{1023} \simeq \frac{419}{1023},\frac{140}{341}$$
$$\frac{210}{1023},\frac{265}{1023} \simeq \frac{137}{341},\frac{412}{1023}$$
$$\frac{211}{1023},\frac{228}{1023} \simeq \frac{377}{1023},\frac{34}{93}$$
$$\frac{212}{1023},\frac{227}{1023} \simeq \frac{130}{341},\frac{131}{341}$$
$$\frac{213}{1023},\frac{214}{1023} \simeq \frac{383}{1023},\frac{128}{341}$$

$$\frac{215}{1023}, \frac{216}{1023} \simeq \frac{127}{341}, \frac{386}{1023}$$
$$\frac{217}{1023}, \frac{226}{1023} \simeq \frac{395}{1023}, \frac{364}{1023}$$
$$\frac{218}{1023}, \frac{225}{1023} \simeq \frac{371}{1023}, \frac{4}{11}$$
$$\frac{219}{1023}, \frac{220}{1023} \simeq \frac{123}{341}, \frac{122}{341}$$
$$\frac{221}{1023}, \frac{222}{1023} \simeq \frac{391}{1023}, \frac{392}{1023}$$
$$\frac{223}{1023}, \frac{224}{1023} \simeq \frac{394}{1023}, \frac{373}{1023}$$
$$\frac{229}{1023}, \frac{230}{1023} \simeq \frac{431}{1023}, \frac{144}{341}$$
$$\frac{232}{1023}, \frac{263}{1023} \simeq \frac{14}{33}, \frac{19}{33}$$
$$\frac{233}{1023}, \frac{242}{1023} \simeq \frac{571}{1023}, \frac{580}{1023}$$
$$\frac{234}{1023}, \frac{241}{1023} \simeq \frac{52}{93}, \frac{193}{341}$$
$$\frac{235}{1023}, \frac{236}{1023} \simeq \frac{574}{1023}, \frac{577}{1023}$$
$$\frac{237}{1023}, \frac{238}{1023} \simeq \frac{189}{341}, \frac{568}{1023}$$
$$\frac{239}{1023}, \frac{240}{1023} \simeq \frac{581}{1023}, \frac{190}{341}$$
$$\frac{243}{1023}, \frac{260}{1023} \simeq \frac{601}{1023}, \frac{422}{1023}$$
$$\frac{244}{1023}, \frac{259}{1023} \simeq \frac{598}{1023}, \frac{425}{1023}$$
$$\frac{245}{1023}, \frac{246}{1023} \simeq \frac{415}{1023}, \frac{416}{1023}$$
$$\frac{247}{1023}, \frac{248}{1023} \simeq \frac{38}{93}, \frac{413}{1023}$$
$$\frac{249}{1023}, \frac{258}{1023} \simeq \frac{428}{1023}, \frac{427}{1023}$$

$$\frac{250}{1023}, \frac{257}{1023} \simeq \frac{596}{1023}, \frac{595}{1023}$$
$$\frac{251}{1023}, \frac{252}{1023} \simeq \frac{593}{1023}, \frac{590}{1023}$$
$$\frac{253}{1023}, \frac{254}{1023} \simeq \frac{424}{1023}, \frac{141}{341}$$
$$\frac{255}{1023}, \frac{256}{1023} \simeq \frac{199}{341}, \frac{142}{341}$$
$$\frac{261}{1023}, \frac{262}{1023} \simeq \frac{197}{341}, \frac{592}{1023}$$
$$\frac{267}{1023}, \frac{268}{1023} \simeq \frac{139}{341}, \frac{138}{341}$$
$$\frac{269}{1023}, \frac{270}{1023} \simeq \frac{599}{1023}, \frac{200}{341}$$
$$\frac{271}{1023}, \frac{272}{1023} \simeq \frac{613}{1023}, \frac{602}{1023}$$
$$\frac{275}{1023}, \frac{276}{1023} \simeq \frac{569}{1023}, \frac{194}{341}$$
$$\frac{277}{1023}, \frac{278}{1023} \simeq \frac{575}{1023}, \frac{192}{341}$$
$$\frac{279}{1023}, \frac{280}{1023} \simeq \frac{191}{341}, \frac{578}{1023}$$
$$\frac{281}{1023}, \frac{290}{1023} \simeq \frac{587}{1023}, \frac{196}{341}$$
$$\frac{282}{1023}, \frac{289}{1023} \simeq \frac{145}{341}, \frac{436}{1023}$$
$$\frac{283}{1023}, \frac{284}{1023} \simeq \frac{433}{1023}, \frac{430}{1023}$$
$$\frac{285}{1023}, \frac{286}{1023} \simeq \frac{53}{93}, \frac{584}{1023}$$
$$\frac{287}{1023}, \frac{288}{1023} \simeq \frac{586}{1023}, \frac{437}{1023}$$
$$\frac{291}{1023}, \frac{292}{1023} \simeq \frac{195}{341}, \frac{146}{341}$$

B.8 Period 11

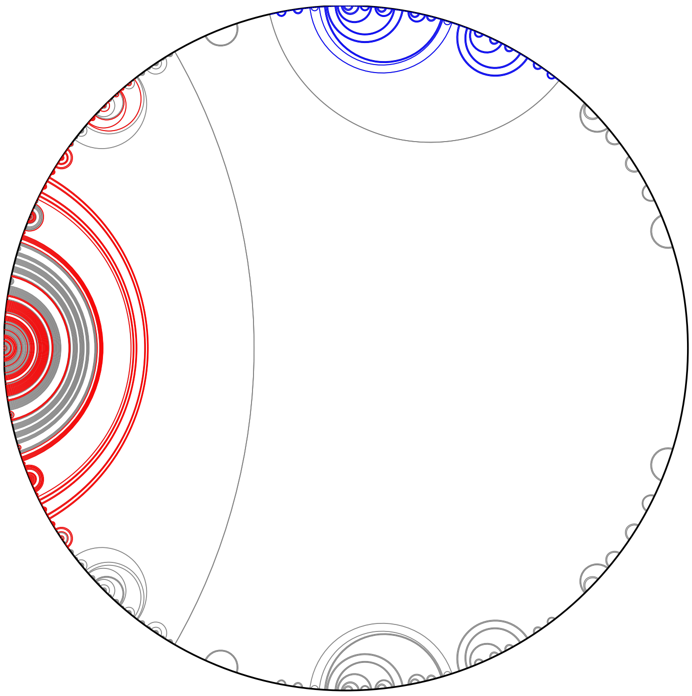

Figure B.8: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 11.

$$\frac{293}{2047}, \frac{294}{2047} \simeq \frac{879}{2047}, \frac{880}{2047}$$
$$\frac{295}{2047}, \frac{296}{2047} \simeq \frac{909}{2047}, \frac{882}{2047}$$
$$\frac{297}{2047}, \frac{306}{2047} \simeq \frac{891}{2047}, \frac{900}{2047}$$
$$\frac{298}{2047}, \frac{305}{2047} \simeq \frac{892}{2047}, \frac{899}{2047}$$
$$\frac{299}{2047}, \frac{300}{2047} \simeq \frac{894}{2047}, \frac{39}{89}$$
$$\frac{301}{2047}, \frac{302}{2047} \simeq \frac{887}{2047}, \frac{888}{2047}$$
$$\frac{303}{2047}, \frac{304}{2047} \simeq \frac{901}{2047}, \frac{10}{23}$$
$$\frac{307}{2047}, \frac{308}{2047} \simeq \frac{921}{2047}, \frac{1126}{2047}$$
$$\frac{309}{2047}, \frac{310}{2047} \simeq \frac{1119}{2047}, \frac{1120}{2047}$$
$$\frac{311}{2047}, \frac{312}{2047} \simeq \frac{1117}{2047}, \frac{1122}{2047}$$
$$\frac{313}{2047}, \frac{322}{2047} \simeq \frac{1131}{2047}, \frac{916}{2047}$$
$$\frac{314}{2047}, \frac{321}{2047} \simeq \frac{915}{2047}, \frac{1132}{2047}$$
$$\frac{315}{2047}, \frac{316}{2047} \simeq \frac{913}{2047}, \frac{910}{2047}$$
$$\frac{317}{2047}, \frac{318}{2047} \simeq \frac{49}{89}, \frac{1128}{2047}$$
$$\frac{319}{2047}, \frac{320}{2047} \simeq \frac{1130}{2047}, \frac{917}{2047}$$
$$\frac{323}{2047}, \frac{324}{2047} \simeq \frac{1129}{2047}, \frac{918}{2047}$$
$$\frac{325}{2047}, \frac{326}{2047} \simeq \frac{911}{2047}, \frac{912}{2047}$$
$$\frac{327}{2047}, \frac{328}{2047} \simeq \frac{1133}{2047}, \frac{914}{2047}$$

$$\frac{329}{2047}, \frac{402}{2047} \simeq \frac{987}{2047}, \frac{1060}{2047}$$
$$\frac{330}{2047}, \frac{401}{2047} \simeq \frac{988}{2047}, \frac{1059}{2047}$$
$$\frac{331}{2047}, \frac{332}{2047} \simeq \frac{990}{2047}, \frac{1057}{2047}$$
$$\frac{333}{2047}, \frac{334}{2047} \simeq \frac{1000}{2047}, \frac{1047}{2047}$$
$$\frac{335}{2047}, \frac{336}{2047} \simeq \frac{997}{2047}, \frac{1050}{2047}$$
$$\frac{337}{2047}, \frac{394}{2047} \simeq \frac{995}{2047}, \frac{1052}{2047}$$
$$\frac{338}{2047}, \frac{393}{2047} \simeq \frac{996}{2047}, \frac{1051}{2047}$$
$$\frac{339}{2047}, \frac{340}{2047} \simeq \frac{1017}{2047}, \frac{1030}{2047}$$
$$\frac{341}{2047}, \frac{342}{2047} \simeq \frac{1024}{2047}, \frac{1023}{2047}$$
$$\frac{343}{2047}, \frac{344}{2047} \simeq \frac{1021}{2047}, \frac{1026}{2047}$$
$$\frac{345}{2047}, \frac{354}{2047} \simeq \frac{44}{89}, \frac{45}{89}$$
$$\frac{346}{2047}, \frac{353}{2047} \simeq \frac{1011}{2047}, \frac{1036}{2047}$$
$$\frac{347}{2047}, \frac{348}{2047} \simeq \frac{1006}{2047}, \frac{1041}{2047}$$
$$\frac{349}{2047}, \frac{350}{2047} \simeq \frac{1016}{2047}, \frac{1031}{2047}$$
$$\frac{351}{2047}, \frac{352}{2047} \simeq \frac{1013}{2047}, \frac{1034}{2047}$$
$$\frac{355}{2047}, \frac{356}{2047} \simeq \frac{1014}{2047}, \frac{1033}{2047}$$
$$\frac{357}{2047}, \frac{390}{2047} \simeq \frac{975}{2047}, \frac{976}{2047}$$
$$\frac{358}{2047}, \frac{389}{2047} \simeq \frac{1071}{2047}, \frac{1072}{2047}$$

$$\frac{359}{2047},\frac{360}{2047} \simeq \frac{973}{2047},\frac{1074}{2047}$$
$$\frac{361}{2047},\frac{370}{2047} \simeq \frac{955}{2047},\frac{1092}{2047}$$
$$\frac{362}{2047},\frac{369}{2047} \simeq \frac{1084}{2047},\frac{1091}{2047}$$
$$\frac{363}{2047},\frac{364}{2047} \simeq \frac{1086}{2047},\frac{1089}{2047}$$
$$\frac{365}{2047},\frac{366}{2047} \simeq \frac{1095}{2047},\frac{1096}{2047}$$
$$\frac{367}{2047},\frac{368}{2047} \simeq \frac{1093}{2047},\frac{954}{2047}$$
$$\frac{371}{2047},\frac{372}{2047} \simeq \frac{985}{2047},\frac{1062}{2047}$$
$$\frac{373}{2047},\frac{374}{2047} \simeq \frac{991}{2047},\frac{992}{2047}$$
$$\frac{375}{2047},\frac{376}{2047} \simeq \frac{1053}{2047},\frac{994}{2047}$$
$$\frac{377}{2047},\frac{386}{2047} \simeq \frac{1067}{2047},\frac{980}{2047}$$
$$\frac{378}{2047},\frac{385}{2047} \simeq \frac{11}{23},\frac{12}{23}$$
$$\frac{379}{2047},\frac{380}{2047} \simeq \frac{1073}{2047},\frac{974}{2047}$$
$$\frac{381}{2047},\frac{382}{2047} \simeq \frac{1063}{2047},\frac{984}{2047}$$
$$\frac{383}{2047},\frac{384}{2047} \simeq \frac{1066}{2047},\frac{981}{2047}$$
$$\frac{387}{2047},\frac{388}{2047} \simeq \frac{1065}{2047},\frac{982}{2047}$$
$$\frac{391}{2047},\frac{392}{2047} \simeq \frac{1069}{2047},\frac{978}{2047}$$
$$\frac{395}{2047},\frac{396}{2047} \simeq \frac{993}{2047},\frac{1054}{2047}$$
$$\frac{397}{2047},\frac{398}{2047} \simeq \frac{983}{2047},\frac{1064}{2047}$$

$$\frac{399}{2047},\frac{400}{2047} \simeq \frac{1061}{2047},\frac{986}{2047}$$
$$\frac{403}{2047},\frac{404}{2047} \simeq \frac{1209}{2047},\frac{1222}{2047}$$
$$\frac{405}{2047},\frac{406}{2047} \simeq \frac{1216}{2047},\frac{1215}{2047}$$
$$\frac{407}{2047},\frac{408}{2047} \simeq \frac{1213}{2047},\frac{1218}{2047}$$
$$\frac{409}{2047},\frac{546}{2047} \simeq \frac{1227}{2047},\frac{1228}{2047}$$
$$\frac{410}{2047},\frac{417}{2047} \simeq \frac{819}{2047},\frac{820}{2047}$$
$$\frac{411}{2047},\frac{412}{2047} \simeq \frac{817}{2047},\frac{814}{2047}$$
$$\frac{413}{2047},\frac{414}{2047} \simeq \frac{839}{2047},\frac{840}{2047}$$
$$\frac{415}{2047},\frac{416}{2047} \simeq \frac{842}{2047},\frac{821}{2047}$$
$$\frac{418}{2047},\frac{537}{2047} \simeq \frac{843}{2047},\frac{844}{2047}$$
$$\frac{419}{2047},\frac{532}{2047} \simeq \frac{822}{2047},\frac{841}{2047}$$
$$\frac{420}{2047},\frac{531}{2047} \simeq \frac{825}{2047},\frac{838}{2047}$$
$$\frac{421}{2047},\frac{422}{2047} \simeq \frac{751}{2047},\frac{752}{2047}$$
$$\frac{423}{2047},\frac{424}{2047} \simeq \frac{781}{2047},\frac{786}{2047}$$
$$\frac{425}{2047},\frac{434}{2047} \simeq \frac{763}{2047},\frac{764}{2047}$$
$$\frac{426}{2047},\frac{433}{2047} \simeq \frac{771}{2047},\frac{772}{2047}$$
$$\frac{427}{2047},\frac{428}{2047} \simeq \frac{766}{2047},\frac{769}{2047}$$
$$\frac{429}{2047},\frac{430}{2047} \simeq \frac{33}{89},\frac{760}{2047}$$

$$\frac{431}{2047}, \frac{432}{2047} \simeq \frac{773}{2047}, \frac{762}{2047}$$
$$\frac{435}{2047}, \frac{452}{2047} \simeq \frac{790}{2047}, \frac{729}{2047}$$
$$\frac{436}{2047}, \frac{451}{2047} \simeq \frac{745}{2047}, \frac{742}{2047}$$
$$\frac{437}{2047}, \frac{438}{2047} \simeq \frac{735}{2047}, \frac{32}{89}$$
$$\frac{439}{2047}, \frac{440}{2047} \simeq \frac{738}{2047}, \frac{733}{2047}$$
$$\frac{441}{2047}, \frac{450}{2047} \simeq \frac{780}{2047}, \frac{747}{2047}$$
$$\frac{442}{2047}, \frac{449}{2047} \simeq \frac{787}{2047}, \frac{788}{2047}$$
$$\frac{443}{2047}, \frac{444}{2047} \simeq \frac{785}{2047}, \frac{34}{89}$$
$$\frac{445}{2047}, \frac{446}{2047} \simeq \frac{744}{2047}, \frac{743}{2047}$$
$$\frac{447}{2047}, \frac{448}{2047} \simeq \frac{789}{2047}, \frac{746}{2047}$$
$$\frac{453}{2047}, \frac{454}{2047} \simeq \frac{783}{2047}, \frac{784}{2047}$$
$$\frac{455}{2047}, \frac{456}{2047} \simeq \frac{754}{2047}, \frac{749}{2047}$$
$$\frac{457}{2047}, \frac{530}{2047} \simeq \frac{860}{2047}, \frac{859}{2047}$$
$$\frac{458}{2047}, \frac{529}{2047} \simeq \frac{867}{2047}, \frac{868}{2047}$$
$$\frac{459}{2047}, \frac{460}{2047} \simeq \frac{862}{2047}, \frac{865}{2047}$$
$$\frac{461}{2047}, \frac{462}{2047} \simeq \frac{1175}{2047}, \frac{1176}{2047}$$
$$\frac{463}{2047}, \frac{464}{2047} \simeq \frac{1189}{2047}, \frac{858}{2047}$$
$$\frac{465}{2047}, \frac{522}{2047} \simeq \frac{1187}{2047}, \frac{1188}{2047}$$
$$\frac{466}{2047}, \frac{521}{2047} \simeq \frac{1179}{2047}, \frac{1180}{2047}$$
$$\frac{467}{2047}, \frac{484}{2047} \simeq \frac{1145}{2047}, \frac{1158}{2047}$$
$$\frac{468}{2047}, \frac{483}{2047} \simeq \frac{1142}{2047}, \frac{1161}{2047}$$
$$\frac{469}{2047}, \frac{470}{2047} \simeq \frac{1151}{2047}, \frac{1152}{2047}$$
$$\frac{471}{2047}, \frac{472}{2047} \simeq \frac{1149}{2047}, \frac{1154}{2047}$$
$$\frac{473}{2047}, \frac{482}{2047} \simeq \frac{1163}{2047}, \frac{1140}{2047}$$
$$\frac{474}{2047}, \frac{481}{2047} \simeq \frac{1139}{2047}, \frac{1164}{2047}$$
$$\frac{475}{2047}, \frac{476}{2047} \simeq \frac{1137}{2047}, \frac{1134}{2047}$$
$$\frac{477}{2047}, \frac{478}{2047} \simeq \frac{1159}{2047}, \frac{1160}{2047}$$
$$\frac{479}{2047}, \frac{480}{2047} \simeq \frac{1162}{2047}, \frac{1141}{2047}$$
$$\frac{485}{2047}, \frac{486}{2047} \simeq \frac{1199}{2047}, \frac{1200}{2047}$$
$$\frac{487}{2047}, \frac{520}{2047} \simeq \frac{850}{2047}, \frac{845}{2047}$$
$$\frac{488}{2047}, \frac{519}{2047} \simeq \frac{1202}{2047}, \frac{1197}{2047}$$
$$\frac{489}{2047}, \frac{498}{2047} \simeq \frac{36}{89}, \frac{827}{2047}$$
$$\frac{490}{2047}, \frac{497}{2047} \simeq \frac{835}{2047}, \frac{836}{2047}$$
$$\frac{491}{2047}, \frac{492}{2047} \simeq \frac{833}{2047}, \frac{830}{2047}$$
$$\frac{493}{2047}, \frac{494}{2047} \simeq \frac{824}{2047}, \frac{823}{2047}$$
$$\frac{495}{2047}, \frac{496}{2047} \simeq \frac{837}{2047}, \frac{826}{2047}$$

$$\frac{499}{2047}, \frac{516}{2047} \simeq \frac{1190}{2047}, \frac{857}{2047}$$
$$\frac{500}{2047}, \frac{515}{2047} \simeq \frac{1193}{2047}, \frac{854}{2047}$$
$$\frac{501}{2047}, \frac{502}{2047} \simeq \frac{1183}{2047}, \frac{1184}{2047}$$
$$\frac{503}{2047}, \frac{504}{2047} \simeq \frac{1186}{2047}, \frac{1181}{2047}$$
$$\frac{505}{2047}, \frac{514}{2047} \simeq \frac{52}{89}, \frac{1195}{2047}$$
$$\frac{506}{2047}, \frac{513}{2047} \simeq \frac{852}{2047}, \frac{37}{89}$$
$$\frac{507}{2047}, \frac{508}{2047} \simeq \frac{849}{2047}, \frac{846}{2047}$$
$$\frac{509}{2047}, \frac{510}{2047} \simeq \frac{1192}{2047}, \frac{1191}{2047}$$
$$\frac{511}{2047}, \frac{512}{2047} \simeq \frac{1194}{2047}, \frac{853}{2047}$$
$$\frac{517}{2047}, \frac{518}{2047} \simeq \frac{848}{2047}, \frac{847}{2047}$$
$$\frac{523}{2047}, \frac{524}{2047} \simeq \frac{1185}{2047}, \frac{1182}{2047}$$
$$\frac{525}{2047}, \frac{526}{2047} \simeq \frac{856}{2047}, \frac{855}{2047}$$
$$\frac{527}{2047}, \frac{528}{2047} \simeq \frac{1178}{2047}, \frac{869}{2047}$$
$$\frac{533}{2047}, \frac{534}{2047} \simeq \frac{831}{2047}, \frac{832}{2047}$$
$$\frac{535}{2047}, \frac{536}{2047} \simeq \frac{834}{2047}, \frac{829}{2047}$$
$$\frac{538}{2047}, \frac{545}{2047} \simeq \frac{1203}{2047}, \frac{1204}{2047}$$
$$\frac{539}{2047}, \frac{540}{2047} \simeq \frac{1201}{2047}, \frac{1198}{2047}$$
$$\frac{541}{2047}, \frac{542}{2047} \simeq \frac{1223}{2047}, \frac{1224}{2047}$$

$$\frac{543}{2047}, \frac{544}{2047} \simeq \frac{1226}{2047}, \frac{1205}{2047}$$
$$\frac{547}{2047}, \frac{548}{2047} \simeq \frac{1225}{2047}, \frac{1206}{2047}$$
$$\frac{549}{2047}, \frac{550}{2047} \simeq \frac{1135}{2047}, \frac{1136}{2047}$$
$$\frac{551}{2047}, \frac{552}{2047} \simeq \frac{1165}{2047}, \frac{1138}{2047}$$
$$\frac{553}{2047}, \frac{562}{2047} \simeq \frac{1147}{2047}, \frac{1156}{2047}$$
$$\frac{554}{2047}, \frac{561}{2047} \simeq \frac{1148}{2047}, \frac{1155}{2047}$$
$$\frac{555}{2047}, \frac{556}{2047} \simeq \frac{50}{89}, \frac{1153}{2047}$$
$$\frac{557}{2047}, \frac{558}{2047} \simeq \frac{1143}{2047}, \frac{1144}{2047}$$
$$\frac{559}{2047}, \frac{560}{2047} \simeq \frac{13}{23}, \frac{1146}{2047}$$
$$\frac{563}{2047}, \frac{564}{2047} \simeq \frac{1177}{2047}, \frac{870}{2047}$$
$$\frac{565}{2047}, \frac{566}{2047} \simeq \frac{863}{2047}, \frac{864}{2047}$$
$$\frac{567}{2047}, \frac{568}{2047} \simeq \frac{866}{2047}, \frac{861}{2047}$$
$$\frac{569}{2047}, \frac{578}{2047} \simeq \frac{876}{2047}, \frac{875}{2047}$$
$$\frac{570}{2047}, \frac{577}{2047} \simeq \frac{1171}{2047}, \frac{1172}{2047}$$
$$\frac{571}{2047}, \frac{572}{2047} \simeq \frac{1169}{2047}, \frac{1166}{2047}$$
$$\frac{573}{2047}, \frac{574}{2047} \simeq \frac{872}{2047}, \frac{871}{2047}$$
$$\frac{575}{2047}, \frac{576}{2047} \simeq \frac{51}{89}, \frac{38}{89}$$
$$\frac{579}{2047}, \frac{580}{2047} \simeq \frac{1174}{2047}, \frac{873}{2047}$$

$$\begin{aligned}
\frac{581}{2047},\frac{582}{2047} &\simeq \frac{1167}{2047},\frac{1168}{2047}\\
\frac{583}{2047},\frac{584}{2047} &\simeq \frac{1170}{2047},\frac{877}{2047}
\end{aligned}$$

B.9 Period 12

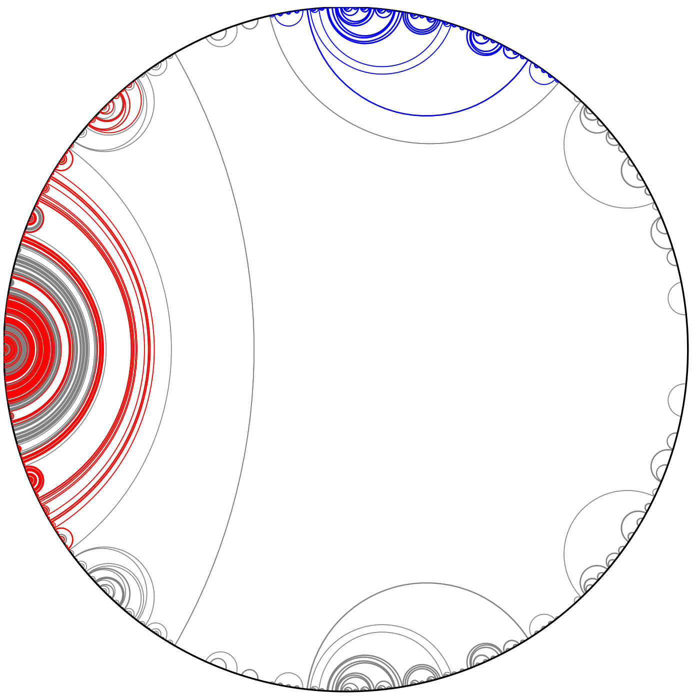

Figure B.9: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 12.

$$\frac{586}{4095}, \frac{593}{4095} \simeq \frac{1756}{4095}, \frac{1763}{4095}$$
$$\frac{587}{4095}, \frac{588}{4095} \simeq \frac{586}{1365}, \frac{587}{1365}$$
$$\frac{589}{4095}, \frac{590}{4095} \simeq \frac{121}{273}, \frac{1816}{4095}$$
$$\frac{591}{4095}, \frac{592}{4095} \simeq \frac{202}{455}, \frac{353}{819}$$
$$\frac{594}{4095}, \frac{649}{4095} \simeq \frac{28}{65}, \frac{1819}{4095}$$
$$\frac{595}{4095}, \frac{596}{4095} \simeq \frac{17}{39}, \frac{1798}{4095}$$
$$\frac{597}{4095}, \frac{598}{4095} \simeq \frac{199}{455}, \frac{256}{585}$$
$$\frac{599}{4095}, \frac{600}{4095} \simeq \frac{1789}{4095}, \frac{46}{105}$$
$$\frac{601}{4095}, \frac{610}{4095} \simeq \frac{356}{819}, \frac{601}{1365}$$
$$\frac{602}{4095}, \frac{609}{4095} \simeq \frac{593}{1365}, \frac{1804}{4095}$$
$$\frac{603}{4095}, \frac{604}{4095} \simeq \frac{1774}{4095}, \frac{1777}{4095}$$
$$\frac{605}{4095}, \frac{606}{4095} \simeq \frac{257}{585}, \frac{40}{91}$$
$$\frac{607}{4095}, \frac{608}{4095} \simeq \frac{1802}{4095}, \frac{137}{315}$$
$$\frac{611}{4095}, \frac{612}{4095} \simeq \frac{198}{455}, \frac{1801}{4095}$$
$$\frac{613}{4095}, \frac{614}{4095} \simeq \frac{613}{1365}, \frac{368}{819}$$
$$\frac{615}{4095}, \frac{616}{4095} \simeq \frac{751}{1365}, \frac{614}{1365}$$
$$\frac{617}{4095}, \frac{626}{4095} \simeq \frac{149}{273}, \frac{748}{1365}$$
$$\frac{618}{4095}, \frac{625}{4095} \simeq \frac{172}{315}, \frac{2243}{4095}$$

$$\frac{619}{4095}, \frac{620}{4095} \simeq \frac{746}{1365}, \frac{249}{455}$$
$$\frac{621}{4095}, \frac{622}{4095} \simeq \frac{2231}{4095}, \frac{248}{455}$$
$$\frac{623}{4095}, \frac{624}{4095} \simeq \frac{449}{819}, \frac{2234}{4095}$$
$$\frac{627}{4095}, \frac{628}{4095} \simeq \frac{151}{273}, \frac{122}{273}$$
$$\frac{629}{4095}, \frac{630}{4095} \simeq \frac{1823}{4095}, \frac{608}{1365}$$
$$\frac{631}{4095}, \frac{632}{4095} \simeq \frac{1826}{4095}, \frac{607}{1365}$$
$$\frac{633}{4095}, \frac{642}{4095} \simeq \frac{452}{819}, \frac{367}{819}$$
$$\frac{634}{4095}, \frac{641}{4095} \simeq \frac{251}{455}, \frac{204}{455}$$
$$\frac{635}{4095}, \frac{636}{4095} \simeq \frac{2257}{4095}, \frac{322}{585}$$
$$\frac{637}{4095}, \frac{638}{4095} \simeq \frac{1832}{4095}, \frac{1831}{4095}$$
$$\frac{639}{4095}, \frac{640}{4095} \simeq \frac{323}{585}, \frac{262}{585}$$
$$\frac{643}{4095}, \frac{644}{4095} \simeq \frac{58}{105}, \frac{47}{105}$$
$$\frac{645}{4095}, \frac{646}{4095} \simeq \frac{451}{819}, \frac{752}{1365}$$
$$\frac{647}{4095}, \frac{648}{4095} \simeq \frac{2258}{4095}, \frac{1837}{4095}$$
$$\frac{651}{4095}, \frac{652}{4095} \simeq \frac{1822}{4095}, \frac{365}{819}$$
$$\frac{653}{4095}, \frac{654}{4095} \simeq \frac{2263}{4095}, \frac{2264}{4095}$$
$$\frac{655}{4095}, \frac{656}{4095} \simeq \frac{2266}{4095}, \frac{1829}{4095}$$
$$\frac{657}{4095}, \frac{1098}{4095} \simeq \frac{29}{65}, \frac{36}{65}$$

$$\frac{658}{4095}, \frac{1097}{4095} \simeq \frac{1828}{4095}, \frac{2267}{4095}$$
$$\frac{659}{4095}, \frac{660}{4095} \simeq \frac{659}{1365}, \frac{706}{1365}$$
$$\frac{661}{4095}, \frac{662}{4095} \simeq \frac{2111}{4095}, \frac{704}{1365}$$
$$\frac{663}{4095}, \frac{664}{4095} \simeq \frac{283}{585}, \frac{302}{585}$$
$$\frac{665}{4095}, \frac{674}{4095} \simeq \frac{19}{39}, \frac{20}{39}$$
$$\frac{666}{4095}, \frac{673}{4095} \simeq \frac{1996}{4095}, \frac{2099}{4095}$$
$$\frac{667}{4095}, \frac{668}{4095} \simeq \frac{667}{1365}, \frac{698}{1365}$$
$$\frac{669}{4095}, \frac{670}{4095} \simeq \frac{1991}{4095}, \frac{2104}{4095}$$
$$\frac{671}{4095}, \frac{672}{4095} \simeq \frac{2101}{4095}, \frac{1994}{4095}$$
$$\frac{675}{4095}, \frac{676}{4095} \simeq \frac{1993}{4095}, \frac{2102}{4095}$$
$$\frac{677}{4095}, \frac{710}{4095} \simeq \frac{2032}{4095}, \frac{2063}{4095}$$
$$\frac{678}{4095}, \frac{709}{4095} \simeq \frac{677}{1365}, \frac{688}{1365}$$
$$\frac{679}{4095}, \frac{680}{4095} \simeq \frac{226}{455}, \frac{229}{455}$$
$$\frac{681}{4095}, \frac{690}{4095} \simeq \frac{227}{455}, \frac{228}{455}$$
$$\frac{682}{4095}, \frac{689}{4095} \simeq \frac{292}{585}, \frac{293}{585}$$
$$\frac{683}{4095}, \frac{684}{4095} \simeq \frac{682}{1365}, \frac{683}{1365}$$
$$\frac{685}{4095}, \frac{686}{4095} \simeq \frac{2039}{4095}, \frac{2056}{4095}$$
$$\frac{687}{4095}, \frac{688}{4095} \simeq \frac{2042}{4095}, \frac{2053}{4095}$$

$$\frac{691}{4095}, \frac{692}{4095} \simeq \frac{674}{1365}, \frac{691}{1365}$$
$$\frac{693}{4095}, \frac{694}{4095} \simeq \frac{33}{65}, \frac{32}{65}$$
$$\frac{695}{4095}, \frac{696}{4095} \simeq \frac{671}{1365}, \frac{694}{1365}$$
$$\frac{697}{4095}, \frac{706}{4095} \simeq \frac{2027}{4095}, \frac{2068}{4095}$$
$$\frac{698}{4095}, \frac{705}{4095} \simeq \frac{52}{105}, \frac{53}{105}$$
$$\frac{699}{4095}, \frac{700}{4095} \simeq \frac{2033}{4095}, \frac{2062}{4095}$$
$$\frac{701}{4095}, \frac{702}{4095} \simeq \frac{289}{585}, \frac{296}{585}$$
$$\frac{703}{4095}, \frac{704}{4095} \simeq \frac{2069}{4095}, \frac{2026}{4095}$$
$$\frac{707}{4095}, \frac{708}{4095} \simeq \frac{45}{91}, \frac{46}{91}$$
$$\frac{711}{4095}, \frac{712}{4095} \simeq \frac{2029}{4095}, \frac{2066}{4095}$$
$$\frac{713}{4095}, \frac{786}{4095} \simeq \frac{652}{1365}, \frac{713}{1365}$$
$$\frac{714}{4095}, \frac{785}{4095} \simeq \frac{391}{819}, \frac{428}{819}$$
$$\frac{716}{4095}, \frac{779}{4095} \simeq \frac{31}{65}, \frac{34}{65}$$
$$\frac{717}{4095}, \frac{718}{4095} \simeq \frac{1943}{4095}, \frac{2152}{4095}$$
$$\frac{719}{4095}, \frac{720}{4095} \simeq \frac{307}{585}, \frac{278}{585}$$
$$\frac{721}{4095}, \frac{778}{4095} \simeq \frac{1948}{4095}, \frac{2147}{4095}$$
$$\frac{722}{4095}, \frac{777}{4095} \simeq \frac{649}{1365}, \frac{716}{1365}$$
$$\frac{723}{4095}, \frac{724}{4095} \simeq \frac{241}{455}, \frac{2182}{4095}$$

$$\frac{725}{4095}, \frac{726}{4095} \simeq \frac{145}{273}, \frac{2176}{4095}$$
$$\frac{727}{4095}, \frac{728}{4095} \simeq \frac{2173}{4095}, \frac{242}{455}$$
$$\frac{729}{4095}, \frac{738}{4095} \simeq \frac{212}{455}, \frac{243}{455}$$
$$\frac{730}{4095}, \frac{737}{4095} \simeq \frac{1907}{4095}, \frac{2188}{4095}$$
$$\frac{731}{4095}, \frac{732}{4095} \simeq \frac{634}{1365}, \frac{731}{1365}$$
$$\frac{733}{4095}, \frac{734}{4095} \simeq \frac{2167}{4095}, \frac{1928}{4095}$$
$$\frac{735}{4095}, \frac{736}{4095} \simeq \frac{2186}{4095}, \frac{1909}{4095}$$
$$\frac{739}{4095}, \frac{740}{4095} \simeq \frac{382}{819}, \frac{437}{819}$$
$$\frac{741}{4095}, \frac{774}{4095} \simeq \frac{709}{1365}, \frac{304}{585}$$
$$\frac{742}{4095}, \frac{773}{4095} \simeq \frac{281}{585}, \frac{656}{1365}$$
$$\frac{743}{4095}, \frac{744}{4095} \simeq \frac{425}{819}, \frac{394}{819}$$
$$\frac{745}{4095}, \frac{754}{4095} \simeq \frac{284}{585}, \frac{301}{585}$$
$$\frac{746}{4095}, \frac{753}{4095} \simeq \frac{1987}{4095}, \frac{2108}{4095}$$
$$\frac{747}{4095}, \frac{748}{4095} \simeq \frac{1982}{4095}, \frac{397}{819}$$
$$\frac{749}{4095}, \frac{750}{4095} \simeq \frac{664}{1365}, \frac{701}{1365}$$
$$\frac{751}{4095}, \frac{752}{4095} \simeq \frac{18}{35}, \frac{17}{35}$$
$$\frac{755}{4095}, \frac{756}{4095} \simeq \frac{2137}{4095}, \frac{1958}{4095}$$
$$\frac{757}{4095}, \frac{758}{4095} \simeq \frac{2143}{4095}, \frac{2144}{4095}$$

$$\frac{759}{4095}, \frac{760}{4095} \simeq \frac{2146}{4095}, \frac{1949}{4095}$$
$$\frac{761}{4095}, \frac{770}{4095} \simeq \frac{164}{315}, \frac{151}{315}$$
$$\frac{762}{4095}, \frac{769}{4095} \simeq \frac{2131}{4095}, \frac{1964}{4095}$$
$$\frac{763}{4095}, \frac{764}{4095} \simeq \frac{2126}{4095}, \frac{1969}{4095}$$
$$\frac{765}{4095}, \frac{766}{4095} \simeq \frac{712}{1365}, \frac{653}{1365}$$
$$\frac{767}{4095}, \frac{768}{4095} \simeq \frac{237}{455}, \frac{218}{455}$$
$$\frac{771}{4095}, \frac{772}{4095} \simeq \frac{2134}{4095}, \frac{1961}{4095}$$
$$\frac{775}{4095}, \frac{776}{4095} \simeq \frac{142}{273}, \frac{131}{273}$$
$$\frac{780}{4095}, \frac{715}{4095} \simeq \frac{10}{21}, \frac{31}{65}$$
$$\frac{781}{4095}, \frac{782}{4095} \simeq \frac{56}{117}, \frac{61}{117}$$
$$\frac{783}{4095}, \frac{784}{4095} \simeq \frac{2138}{4095}, \frac{1957}{4095}$$
$$\frac{787}{4095}, \frac{788}{4095} \simeq \frac{398}{819}, \frac{421}{819}$$
$$\frac{789}{4095}, \frac{790}{4095} \simeq \frac{661}{1365}, \frac{1984}{4095}$$
$$\frac{791}{4095}, \frac{792}{4095} \simeq \frac{662}{1365}, \frac{703}{1365}$$
$$\frac{793}{4095}, \frac{802}{4095} \simeq \frac{1972}{4095}, \frac{2123}{4095}$$
$$\frac{794}{4095}, \frac{801}{4095} \simeq \frac{219}{455}, \frac{236}{455}$$
$$\frac{795}{4095}, \frac{796}{4095} \simeq \frac{1966}{4095}, \frac{2129}{4095}$$
$$\frac{797}{4095}, \frac{798}{4095} \simeq \frac{152}{315}, \frac{163}{315}$$

$$\frac{799}{4095},\frac{800}{4095} \simeq \frac{2122}{4095},\frac{1973}{4095}$$
$$\frac{803}{4095},\frac{804}{4095} \simeq \frac{94}{195},\frac{101}{195}$$
$$\frac{805}{4095},\frac{806}{4095} \simeq \frac{23}{39},\frac{2416}{4095}$$
$$\frac{807}{4095},\frac{808}{4095} \simeq \frac{163}{273},\frac{70}{117}$$
$$\frac{809}{4095},\frac{818}{4095} \simeq \frac{809}{1365},\frac{2428}{4095}$$
$$\frac{810}{4095},\frac{817}{4095} \simeq \frac{487}{819},\frac{116}{195}$$
$$\frac{811}{4095},\frac{812}{4095} \simeq \frac{54}{91},\frac{811}{1365}$$
$$\frac{813}{4095},\frac{814}{4095} \simeq \frac{2423}{4095},\frac{808}{1365}$$
$$\frac{815}{4095},\frac{816}{4095} \simeq \frac{2437}{4095},\frac{2426}{4095}$$
$$\frac{819}{4095},\frac{1092}{4095} \simeq \frac{3}{5},\frac{818}{1365}$$
$$\frac{820}{4095},\frac{835}{4095} \simeq \frac{562}{1365},\frac{547}{1365}$$
$$\frac{821}{4095},\frac{822}{4095} \simeq \frac{233}{585},\frac{544}{1365}$$
$$\frac{823}{4095},\frac{824}{4095} \simeq \frac{1634}{4095},\frac{181}{455}$$
$$\frac{825}{4095},\frac{834}{4095} \simeq \frac{548}{1365},\frac{1643}{4095}$$
$$\frac{826}{4095},\frac{833}{4095} \simeq \frac{187}{455},\frac{1684}{4095}$$
$$\frac{827}{4095},\frac{828}{4095} \simeq \frac{1681}{4095},\frac{1678}{4095}$$
$$\frac{829}{4095},\frac{830}{4095} \simeq \frac{328}{819},\frac{1639}{4095}$$
$$\frac{831}{4095},\frac{832}{4095} \simeq \frac{337}{819},\frac{1642}{4095}$$

$$\frac{836}{4095},\frac{1075}{4095} \simeq \frac{563}{1365},\frac{802}{1365}$$
$$\frac{837}{4095},\frac{838}{4095} \simeq \frac{1679}{4095},\frac{16}{39}$$
$$\frac{839}{4095},\frac{840}{4095} \simeq \frac{110}{273},\frac{47}{117}$$
$$\frac{841}{4095},\frac{914}{4095} \simeq \frac{1499}{4095},\frac{100}{273}$$
$$\frac{842}{4095},\frac{913}{4095} \simeq \frac{1507}{4095},\frac{116}{315}$$
$$\frac{843}{4095},\frac{844}{4095} \simeq \frac{1502}{4095},\frac{43}{117}$$
$$\frac{846}{4095},\frac{909}{4095} \simeq \frac{1496}{4095},\frac{1559}{4095}$$
$$\frac{847}{4095},\frac{848}{4095} \simeq \frac{121}{315},\frac{1562}{4095}$$
$$\frac{849}{4095},\frac{906}{4095} \simeq \frac{1571}{4095},\frac{524}{1365}$$
$$\frac{850}{4095},\frac{905}{4095} \simeq \frac{521}{1365},\frac{1564}{4095}$$
$$\frac{851}{4095},\frac{868}{4095} \simeq \frac{218}{585},\frac{1529}{4095}$$
$$\frac{852}{4095},\frac{867}{4095} \simeq \frac{514}{1365},\frac{103}{273}$$
$$\frac{853}{4095},\frac{854}{4095} \simeq \frac{307}{819},\frac{512}{1365}$$
$$\frac{855}{4095},\frac{856}{4095} \simeq \frac{73}{195},\frac{1538}{4095}$$
$$\frac{857}{4095},\frac{866}{4095} \simeq \frac{1516}{4095},\frac{17}{45}$$
$$\frac{858}{4095},\frac{865}{4095} \simeq \frac{1523}{4095},\frac{508}{1365}$$
$$\frac{859}{4095},\frac{860}{4095} \simeq \frac{506}{1365},\frac{13}{35}$$
$$\frac{861}{4095},\frac{862}{4095} \simeq \frac{1543}{4095},\frac{1544}{4095}$$

$$\frac{863}{4095}, \frac{864}{4095} \simeq \frac{1546}{4095}, \frac{305}{819}$$
$$\frac{869}{4095}, \frac{870}{4095} \simeq \frac{97}{273}, \frac{16}{45}$$
$$\frac{871}{4095}, \frac{872}{4095} \simeq \frac{298}{819}, \frac{33}{91}$$
$$\frac{873}{4095}, \frac{882}{4095} \simeq \frac{163}{455}, \frac{1468}{4095}$$
$$\frac{874}{4095}, \frac{881}{4095} \simeq \frac{295}{819}, \frac{164}{455}$$
$$\frac{875}{4095}, \frac{876}{4095} \simeq \frac{14}{39}, \frac{491}{1365}$$
$$\frac{877}{4095}, \frac{878}{4095} \simeq \frac{209}{585}, \frac{488}{1365}$$
$$\frac{879}{4095}, \frac{880}{4095} \simeq \frac{211}{585}, \frac{1466}{4095}$$
$$\frac{883}{4095}, \frac{900}{4095} \simeq \frac{223}{585}, \frac{166}{455}$$
$$\frac{884}{4095}, \frac{899}{4095} \simeq \frac{1577}{4095}, \frac{1574}{4095}$$
$$\frac{885}{4095}, \frac{886}{4095} \simeq \frac{1567}{4095}, \frac{224}{585}$$
$$\frac{887}{4095}, \frac{888}{4095} \simeq \frac{314}{819}, \frac{313}{819}$$
$$\frac{889}{4095}, \frac{898}{4095} \simeq \frac{1579}{4095}, \frac{212}{585}$$
$$\frac{890}{4095}, \frac{897}{4095} \simeq \frac{1492}{4095}, \frac{71}{195}$$
$$\frac{891}{4095}, \frac{892}{4095} \simeq \frac{1489}{4095}, \frac{1486}{4095}$$
$$\frac{893}{4095}, \frac{894}{4095} \simeq \frac{1576}{4095}, \frac{5}{13}$$
$$\frac{895}{4095}, \frac{896}{4095} \simeq \frac{526}{1365}, \frac{1493}{4095}$$
$$\frac{901}{4095}, \frac{902}{4095} \simeq \frac{1487}{4095}, \frac{496}{1365}$$

$$\frac{903}{4095}, \frac{904}{4095} \simeq \frac{527}{1365}, \frac{162}{455}$$
$$\frac{907}{4095}, \frac{908}{4095} \simeq \frac{174}{455}, \frac{523}{1365}$$
$$\frac{910}{4095}, \frac{845}{4095} \simeq \frac{23}{63}, \frac{1496}{4095}$$
$$\frac{911}{4095}, \frac{912}{4095} \simeq \frac{503}{1365}, \frac{214}{585}$$
$$\frac{915}{4095}, \frac{1060}{4095} \simeq \frac{578}{1365}, \frac{1721}{4095}$$
$$\frac{916}{4095}, \frac{1059}{4095} \simeq \frac{193}{455}, \frac{1718}{4095}$$
$$\frac{917}{4095}, \frac{918}{4095} \simeq \frac{1727}{4095}, \frac{192}{455}$$
$$\frac{919}{4095}, \frac{920}{4095} \simeq \frac{346}{819}, \frac{115}{273}$$
$$\frac{921}{4095}, \frac{1058}{4095} \simeq \frac{116}{273}, \frac{1739}{4095}$$
$$\frac{922}{4095}, \frac{1057}{4095} \simeq \frac{157}{273}, \frac{2356}{4095}$$
$$\frac{923}{4095}, \frac{924}{4095} \simeq \frac{181}{315}, \frac{470}{819}$$
$$\frac{925}{4095}, \frac{926}{4095} \simeq \frac{475}{819}, \frac{264}{455}$$
$$\frac{927}{4095}, \frac{928}{4095} \simeq \frac{2378}{4095}, \frac{1717}{4095}$$
$$\frac{929}{4095}, \frac{1050}{4095} \simeq \frac{44}{105}, \frac{49}{117}$$
$$\frac{930}{4095}, \frac{1049}{4095} \simeq \frac{61}{105}, \frac{68}{117}$$
$$\frac{931}{4095}, \frac{1044}{4095} \simeq \frac{262}{455}, \frac{2377}{4095}$$
$$\frac{932}{4095}, \frac{1043}{4095} \simeq \frac{787}{1365}, \frac{2374}{4095}$$
$$\frac{933}{4095}, \frac{934}{4095} \simeq \frac{2287}{4095}, \frac{176}{315}$$

$$\frac{935}{4095}, \frac{968}{4095} \simeq \frac{331}{585}, \frac{458}{819}$$
$$\frac{936}{4095}, \frac{967}{4095} \simeq \frac{258}{455}, \frac{457}{819}$$
$$\frac{937}{4095}, \frac{946}{4095} \simeq \frac{2299}{4095}, \frac{2308}{4095}$$
$$\frac{938}{4095}, \frac{945}{4095} \simeq \frac{460}{819}, \frac{769}{1365}$$
$$\frac{939}{4095}, \frac{940}{4095} \simeq \frac{2302}{4095}, \frac{461}{819}$$
$$\frac{941}{4095}, \frac{942}{4095} \simeq \frac{51}{91}, \frac{328}{585}$$
$$\frac{943}{4095}, \frac{944}{4095} \simeq \frac{2309}{4095}, \frac{766}{1365}$$
$$\frac{947}{4095}, \frac{964}{4095} \simeq \frac{2329}{4095}, \frac{2278}{4095}$$
$$\frac{948}{4095}, \frac{963}{4095} \simeq \frac{2281}{4095}, \frac{2326}{4095}$$
$$\frac{949}{4095}, \frac{950}{4095} \simeq \frac{757}{1365}, \frac{2272}{4095}$$
$$\frac{951}{4095}, \frac{952}{4095} \simeq \frac{758}{1365}, \frac{2269}{4095}$$
$$\frac{953}{4095}, \frac{962}{4095} \simeq \frac{332}{585}, \frac{761}{1365}$$
$$\frac{954}{4095}, \frac{961}{4095} \simeq \frac{2323}{4095}, \frac{2284}{4095}$$
$$\frac{955}{4095}, \frac{956}{4095} \simeq \frac{2321}{4095}, \frac{2318}{4095}$$
$$\frac{957}{4095}, \frac{958}{4095} \simeq \frac{152}{273}, \frac{2279}{4095}$$
$$\frac{959}{4095}, \frac{960}{4095} \simeq \frac{155}{273}, \frac{326}{585}$$
$$\frac{965}{4095}, \frac{966}{4095} \simeq \frac{773}{1365}, \frac{464}{819}$$
$$\frac{969}{4095}, \frac{1042}{4095} \simeq \frac{2396}{4095}, \frac{479}{819}$$

$$\frac{970}{4095}, \frac{1041}{4095} \simeq \frac{267}{455}, \frac{2404}{4095}$$
$$\frac{971}{4095}, \frac{972}{4095} \simeq \frac{2398}{4095}, \frac{343}{585}$$
$$\frac{973}{4095}, \frac{974}{4095} \simeq \frac{1688}{4095}, \frac{241}{585}$$
$$\frac{975}{4095}, \frac{1040}{4095} \simeq \frac{37}{63}, \frac{38}{65}$$
$$\frac{976}{4095}, \frac{1039}{4095} \simeq \frac{27}{65}, \frac{38}{65}$$
$$\frac{977}{4095}, \frac{1034}{4095} \simeq \frac{340}{819}, \frac{1699}{4095}$$
$$\frac{978}{4095}, \frac{1033}{4095} \simeq \frac{188}{455}, \frac{1691}{4095}$$
$$\frac{979}{4095}, \frac{996}{4095} \simeq \frac{334}{819}, \frac{1657}{4095}$$
$$\frac{980}{4095}, \frac{995}{4095} \simeq \frac{239}{585}, \frac{1654}{4095}$$
$$\frac{981}{4095}, \frac{982}{4095} \simeq \frac{1663}{4095}, \frac{128}{315}$$
$$\frac{983}{4095}, \frac{984}{4095} \simeq \frac{238}{585}, \frac{1661}{4095}$$
$$\frac{985}{4095}, \frac{994}{4095} \simeq \frac{1676}{4095}, \frac{335}{819}$$
$$\frac{986}{4095}, \frac{993}{4095} \simeq \frac{236}{585}, \frac{127}{315}$$
$$\frac{987}{4095}, \frac{988}{4095} \simeq \frac{1649}{4095}, \frac{1646}{4095}$$
$$\frac{989}{4095}, \frac{990}{4095} \simeq \frac{1672}{4095}, \frac{557}{1365}$$
$$\frac{991}{4095}, \frac{992}{4095} \simeq \frac{186}{455}, \frac{551}{1365}$$
$$\frac{997}{4095}, \frac{998}{4095} \simeq \frac{1711}{4095}, \frac{1712}{4095}$$
$$\frac{999}{4095}, \frac{1032}{4095} \simeq \frac{2386}{4095}, \frac{2381}{4095}$$

$$\frac{1000}{4095}, \frac{1031}{4095} \simeq \frac{1714}{4095}, \frac{1709}{4095}$$
$$\frac{1001}{4095}, \frac{1010}{4095} \simeq \frac{788}{1365}, \frac{2363}{4095}$$
$$\frac{1002}{4095}, \frac{1009}{4095} \simeq \frac{2371}{4095}, \frac{2372}{4095}$$
$$\frac{1003}{4095}, \frac{1004}{4095} \simeq \frac{2369}{4095}, \frac{26}{45}$$
$$\frac{1005}{4095}, \frac{1006}{4095} \simeq \frac{472}{819}, \frac{337}{585}$$
$$\frac{1007}{4095}, \frac{1008}{4095} \simeq \frac{113}{195}, \frac{2362}{4095}$$
$$\frac{1011}{4095}, \frac{1028}{4095} \simeq \frac{2393}{4095}, \frac{1702}{4095}$$
$$\frac{1012}{4095}, \frac{1027}{4095} \simeq \frac{478}{819}, \frac{341}{819}$$
$$\frac{1013}{4095}, \frac{1014}{4095} \simeq \frac{1696}{4095}, \frac{113}{273}$$
$$\frac{1015}{4095}, \frac{1016}{4095} \simeq \frac{566}{1365}, \frac{1693}{4095}$$
$$\frac{1017}{4095}, \frac{1026}{4095} \simeq \frac{244}{585}, \frac{569}{1365}$$
$$\frac{1018}{4095}, \frac{1025}{4095} \simeq \frac{796}{1365}, \frac{341}{585}$$
$$\frac{1019}{4095}, \frac{1020}{4095} \simeq \frac{53}{91}, \frac{794}{1365}$$
$$\frac{1021}{4095}, \frac{1022}{4095} \simeq \frac{568}{1365}, \frac{131}{315}$$
$$\frac{1023}{4095}, \frac{1024}{4095} \simeq \frac{1706}{4095}, \frac{2389}{4095}$$
$$\frac{1029}{4095}, \frac{1030}{4095} \simeq \frac{2384}{4095}, \frac{2383}{4095}$$
$$\frac{1035}{4095}, \frac{1036}{4095} \simeq \frac{1697}{4095}, \frac{242}{585}$$
$$\frac{1037}{4095}, \frac{1038}{4095} \simeq \frac{184}{315}, \frac{797}{1365}$$

$$\frac{1045}{4095}, \frac{1046}{4095} \simeq \frac{263}{455}, \frac{2368}{4095}$$
$$\frac{1047}{4095}, \frac{1048}{4095} \simeq \frac{158}{273}, \frac{473}{819}$$
$$\frac{1051}{4095}, \frac{1052}{4095} \simeq \frac{571}{1365}, \frac{38}{91}$$
$$\frac{1053}{4095}, \frac{1054}{4095} \simeq \frac{248}{585}, \frac{347}{819}$$
$$\frac{1055}{4095}, \frac{1056}{4095} \simeq \frac{2357}{4095}, \frac{1738}{4095}$$
$$\frac{1061}{4095}, \frac{1062}{4095} \simeq \frac{183}{455}, \frac{1648}{4095}$$
$$\frac{1063}{4095}, \frac{1064}{4095} \simeq \frac{1682}{4095}, \frac{43}{105}$$
$$\frac{1065}{4095}, \frac{1074}{4095} \simeq \frac{79}{195}, \frac{332}{819}$$
$$\frac{1066}{4095}, \frac{1073}{4095} \simeq \frac{1667}{4095}, \frac{556}{1365}$$
$$\frac{1067}{4095}, \frac{1068}{4095} \simeq \frac{554}{1365}, \frac{37}{91}$$
$$\frac{1069}{4095}, \frac{1070}{4095} \simeq \frac{331}{819}, \frac{184}{455}$$
$$\frac{1071}{4095}, \frac{1072}{4095} \simeq \frac{1669}{4095}, \frac{1658}{4095}$$
$$\frac{1076}{4095}, \frac{1091}{4095} \simeq \frac{818}{1365}, \frac{803}{1365}$$
$$\frac{1077}{4095}, \frac{1078}{4095} \simeq \frac{2399}{4095}, \frac{160}{273}$$
$$\frac{1079}{4095}, \frac{1080}{4095} \simeq \frac{2402}{4095}, \frac{799}{1365}$$
$$\frac{1081}{4095}, \frac{1090}{4095} \simeq \frac{268}{455}, \frac{2411}{4095}$$
$$\frac{1082}{4095}, \frac{1089}{4095} \simeq \frac{817}{1365}, \frac{2452}{4095}$$
$$\frac{1083}{4095}, \frac{1084}{4095} \simeq \frac{2449}{4095}, \frac{2446}{4095}$$

$$\frac{1085}{4095},\frac{1086}{4095} \simeq \frac{344}{585},\frac{2407}{4095}$$
$$\frac{1087}{4095},\frac{1088}{4095} \simeq \frac{2453}{4095},\frac{482}{819}$$
$$\frac{1093}{4095},\frac{1094}{4095} \simeq \frac{2447}{4095},\frac{272}{455}$$
$$\frac{1095}{4095},\frac{1096}{4095} \simeq \frac{62}{105},\frac{2413}{4095}$$
$$\frac{1099}{4095},\frac{1100}{4095} \simeq \frac{454}{819},\frac{2273}{4095}$$
$$\frac{1101}{4095},\frac{1102}{4095} \simeq \frac{179}{315},\frac{776}{1365}$$
$$\frac{1103}{4095},\frac{1104}{4095} \simeq \frac{466}{819},\frac{253}{455}$$
$$\frac{1106}{4095},\frac{1161}{4095} \simeq \frac{2276}{4095},\frac{37}{65}$$
$$\frac{1107}{4095},\frac{1108}{4095} \simeq \frac{2297}{4095},\frac{22}{39}$$
$$\frac{1109}{4095},\frac{1110}{4095} \simeq \frac{329}{585},\frac{256}{455}$$
$$\frac{1111}{4095},\frac{1112}{4095} \simeq \frac{59}{105},\frac{2306}{4095}$$
$$\frac{1113}{4095},\frac{1122}{4095} \simeq \frac{764}{1365},\frac{463}{819}$$
$$\frac{1114}{4095},\frac{1121}{4095} \simeq \frac{2291}{4095},\frac{772}{1365}$$
$$\frac{1115}{4095},\frac{1116}{4095} \simeq \frac{254}{455},\frac{109}{195}$$
$$\frac{1117}{4095},\frac{1118}{4095} \simeq \frac{2311}{4095},\frac{2312}{4095}$$
$$\frac{1119}{4095},\frac{1120}{4095} \simeq \frac{178}{315},\frac{2293}{4095}$$
$$\frac{1123}{4095},\frac{1124}{4095} \simeq \frac{2294}{4095},\frac{257}{455}$$
$$\frac{1125}{4095},\frac{1126}{4095} \simeq \frac{2351}{4095},\frac{112}{195}$$

$$\frac{1127}{4095},\frac{1128}{4095} \simeq \frac{2354}{4095},\frac{1741}{4095}$$
$$\frac{1129}{4095},\frac{1138}{4095} \simeq \frac{1723}{4095},\frac{1724}{4095}$$
$$\frac{1130}{4095},\frac{1137}{4095} \simeq \frac{577}{1365},\frac{1732}{4095}$$
$$\frac{1131}{4095},\frac{1132}{4095} \simeq \frac{1726}{4095},\frac{19}{45}$$
$$\frac{1133}{4095},\frac{1134}{4095} \simeq \frac{191}{455},\frac{344}{819}$$
$$\frac{1135}{4095},\frac{1136}{4095} \simeq \frac{1733}{4095},\frac{82}{195}$$
$$\frac{1139}{4095},\frac{1140}{4095} \simeq \frac{2342}{4095},\frac{1753}{4095}$$
$$\frac{1141}{4095},\frac{1142}{4095} \simeq \frac{467}{819},\frac{2336}{4095}$$
$$\frac{1143}{4095},\frac{1144}{4095} \simeq \frac{334}{585},\frac{2333}{4095}$$
$$\frac{1145}{4095},\frac{1154}{4095} \simeq \frac{2348}{4095},\frac{2347}{4095}$$
$$\frac{1146}{4095},\frac{1153}{4095} \simeq \frac{1748}{4095},\frac{1747}{4095}$$
$$\frac{1147}{4095},\frac{1148}{4095} \simeq \frac{349}{819},\frac{134}{315}$$
$$\frac{1149}{4095},\frac{1150}{4095} \simeq \frac{2344}{4095},\frac{781}{1365}$$
$$\frac{1151}{4095},\frac{1152}{4095} \simeq \frac{782}{1365},\frac{583}{1365}$$
$$\frac{1155}{4095},\frac{1156}{4095} \simeq \frac{67}{117},\frac{50}{117}$$
$$\frac{1157}{4095},\frac{1158}{4095} \simeq \frac{83}{195},\frac{1744}{4095}$$
$$\frac{1159}{4095},\frac{1160}{4095} \simeq \frac{261}{455},\frac{194}{455}$$
$$\frac{1162}{4095},\frac{1169}{4095} \simeq \frac{2332}{4095},\frac{2339}{4095}$$

$$\begin{aligned}
\frac{1163}{4095}, \frac{1164}{4095} &\simeq \frac{778}{1365}, \frac{779}{1365} \\
\frac{1165}{4095}, \frac{1166}{4095} &\simeq \frac{1751}{4095}, \frac{584}{1365} \\
\frac{1167}{4095}, \frac{1168}{4095} &\simeq \frac{2341}{4095}, \frac{1754}{4095}
\end{aligned}$$

B.10 Period 13

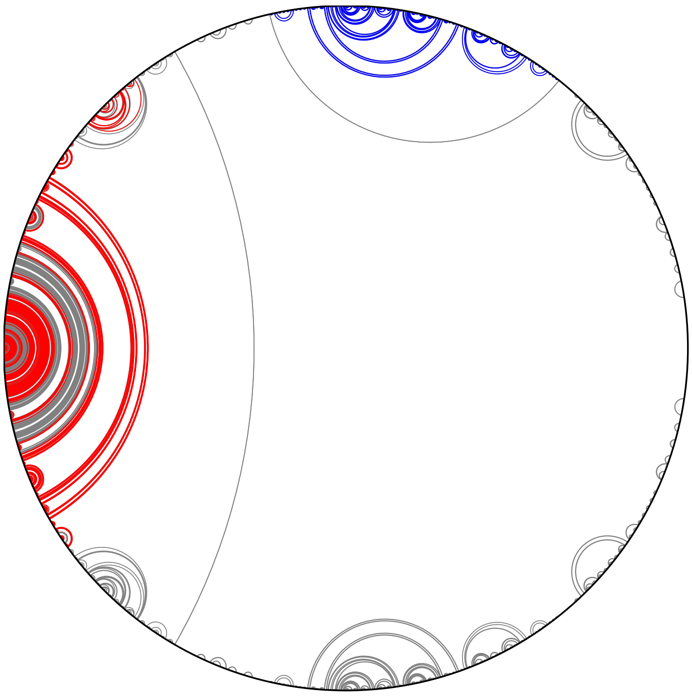

Figure B.10: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 13.

$$\frac{1171}{8191}, \frac{1172}{8191} \simeq \frac{3513}{8191}, \frac{3526}{8191}$$
$$\frac{1173}{8191}, \frac{1174}{8191} \simeq \frac{3519}{8191}, \frac{3520}{8191}$$
$$\frac{1175}{8191}, \frac{1176}{8191} \simeq \frac{3517}{8191}, \frac{3522}{8191}$$
$$\frac{1177}{8191}, \frac{1186}{8191} \simeq \frac{3531}{8191}, \frac{3636}{8191}$$
$$\frac{1178}{8191}, \frac{1185}{8191} \simeq \frac{3532}{8191}, \frac{3635}{8191}$$
$$\frac{1179}{8191}, \frac{1180}{8191} \simeq \frac{3630}{8191}, \frac{3633}{8191}$$
$$\frac{1181}{8191}, \frac{1182}{8191} \simeq \frac{3527}{8191}, \frac{3528}{8191}$$
$$\frac{1183}{8191}, \frac{1184}{8191} \simeq \frac{3637}{8191}, \frac{3530}{8191}$$
$$\frac{1187}{8191}, \frac{1188}{8191} \simeq \frac{3529}{8191}, \frac{3638}{8191}$$
$$\frac{1189}{8191}, \frac{1190}{8191} \simeq \frac{3567}{8191}, \frac{3568}{8191}$$
$$\frac{1191}{8191}, \frac{1192}{8191} \simeq \frac{3570}{8191}, \frac{3597}{8191}$$
$$\frac{1193}{8191}, \frac{1202}{8191} \simeq \frac{3579}{8191}, \frac{3588}{8191}$$
$$\frac{1194}{8191}, \frac{1201}{8191} \simeq \frac{3580}{8191}, \frac{3587}{8191}$$
$$\frac{1195}{8191}, \frac{1196}{8191} \simeq \frac{3582}{8191}, \frac{3585}{8191}$$
$$\frac{1197}{8191}, \frac{1198}{8191} \simeq \frac{3575}{8191}, \frac{3576}{8191}$$
$$\frac{1199}{8191}, \frac{1200}{8191} \simeq \frac{3578}{8191}, \frac{3589}{8191}$$
$$\frac{1203}{8191}, \frac{1204}{8191} \simeq \frac{3558}{8191}, \frac{3609}{8191}$$
$$\frac{1205}{8191}, \frac{1206}{8191} \simeq \frac{3551}{8191}, \frac{3552}{8191}$$
$$\frac{1207}{8191}, \frac{1208}{8191} \simeq \frac{3549}{8191}, \frac{3554}{8191}$$
$$\frac{1209}{8191}, \frac{1218}{8191} \simeq \frac{3563}{8191}, \frac{3604}{8191}$$
$$\frac{1210}{8191}, \frac{1217}{8191} \simeq \frac{3564}{8191}, \frac{3603}{8191}$$
$$\frac{1211}{8191}, \frac{1212}{8191} \simeq \frac{3598}{8191}, \frac{3601}{8191}$$
$$\frac{1213}{8191}, \frac{1214}{8191} \simeq \frac{3559}{8191}, \frac{3560}{8191}$$
$$\frac{1215}{8191}, \frac{1216}{8191} \simeq \frac{3605}{8191}, \frac{3562}{8191}$$
$$\frac{1219}{8191}, \frac{1220}{8191} \simeq \frac{3561}{8191}, \frac{3606}{8191}$$
$$\frac{1221}{8191}, \frac{1222}{8191} \simeq \frac{3599}{8191}, \frac{3600}{8191}$$
$$\frac{1223}{8191}, \frac{1224}{8191} \simeq \frac{3565}{8191}, \frac{3602}{8191}$$
$$\frac{1225}{8191}, \frac{1298}{8191} \simeq \frac{3675}{8191}, \frac{4516}{8191}$$
$$\frac{1226}{8191}, \frac{1233}{8191} \simeq \frac{3683}{8191}, \frac{4508}{8191}$$
$$\frac{1227}{8191}, \frac{1228}{8191} \simeq \frac{3678}{8191}, \frac{3681}{8191}$$
$$\frac{1229}{8191}, \frac{1230}{8191} \simeq \frac{4503}{8191}, \frac{4504}{8191}$$
$$\frac{1231}{8191}, \frac{1232}{8191} \simeq \frac{4506}{8191}, \frac{3685}{8191}$$
$$\frac{1234}{8191}, \frac{1289}{8191} \simeq \frac{3684}{8191}, \frac{4507}{8191}$$
$$\frac{1235}{8191}, \frac{1236}{8191} \simeq \frac{4473}{8191}, \frac{4486}{8191}$$
$$\frac{1237}{8191}, \frac{1238}{8191} \simeq \frac{4479}{8191}, \frac{4480}{8191}$$
$$\frac{1239}{8191}, \frac{1240}{8191} \simeq \frac{4477}{8191}, \frac{4482}{8191}$$

$$\frac{1241}{8191}, \frac{1250}{8191} \simeq \frac{4468}{8191}, \frac{4491}{8191}$$
$$\frac{1242}{8191}, \frac{1249}{8191} \simeq \frac{3724}{8191}, \frac{4467}{8191}$$
$$\frac{1243}{8191}, \frac{1244}{8191} \simeq \frac{4462}{8191}, \frac{3729}{8191}$$
$$\frac{1245}{8191}, \frac{1246}{8191} \simeq \frac{4487}{8191}, \frac{4488}{8191}$$
$$\frac{1247}{8191}, \frac{1248}{8191} \simeq \frac{4490}{8191}, \frac{4469}{8191}$$
$$\frac{1251}{8191}, \frac{1252}{8191} \simeq \frac{4470}{8191}, \frac{4489}{8191}$$
$$\frac{1253}{8191}, \frac{1254}{8191} \simeq \frac{4527}{8191}, \frac{4528}{8191}$$
$$\frac{1255}{8191}, \frac{1256}{8191} \simeq \frac{4530}{8191}, \frac{3661}{8191}$$
$$\frac{1257}{8191}, \frac{1266}{8191} \simeq \frac{3643}{8191}, \frac{3652}{8191}$$
$$\frac{1258}{8191}, \frac{1265}{8191} \simeq \frac{3644}{8191}, \frac{3651}{8191}$$
$$\frac{1259}{8191}, \frac{1260}{8191} \simeq \frac{3646}{8191}, \frac{3649}{8191}$$
$$\frac{1261}{8191}, \frac{1262}{8191} \simeq \frac{3639}{8191}, \frac{3640}{8191}$$
$$\frac{1263}{8191}, \frac{1264}{8191} \simeq \frac{3653}{8191}, \frac{3642}{8191}$$
$$\frac{1267}{8191}, \frac{1268}{8191} \simeq \frac{4518}{8191}, \frac{3673}{8191}$$
$$\frac{1269}{8191}, \frac{1270}{8191} \simeq \frac{4511}{8191}, \frac{4512}{8191}$$
$$\frac{1271}{8191}, \frac{1272}{8191} \simeq \frac{4514}{8191}, \frac{4509}{8191}$$
$$\frac{1273}{8191}, \frac{1282}{8191} \simeq \frac{4523}{8191}, \frac{3668}{8191}$$
$$\frac{1274}{8191}, \frac{1281}{8191} \simeq \frac{4524}{8191}, \frac{3667}{8191}$$
$$\frac{1275}{8191}, \frac{1276}{8191} \simeq \frac{3665}{8191}, \frac{3662}{8191}$$
$$\frac{1277}{8191}, \frac{1278}{8191} \simeq \frac{4520}{8191}, \frac{4519}{8191}$$
$$\frac{1279}{8191}, \frac{1280}{8191} \simeq \frac{4522}{8191}, \frac{3669}{8191}$$
$$\frac{1283}{8191}, \frac{1284}{8191} \simeq \frac{4521}{8191}, \frac{3670}{8191}$$
$$\frac{1285}{8191}, \frac{1286}{8191} \simeq \frac{3663}{8191}, \frac{3664}{8191}$$
$$\frac{1287}{8191}, \frac{1288}{8191} \simeq \frac{4525}{8191}, \frac{3666}{8191}$$
$$\frac{1290}{8191}, \frac{1297}{8191} \simeq \frac{3676}{8191}, \frac{4515}{8191}$$
$$\frac{1291}{8191}, \frac{1292}{8191} \simeq \frac{4510}{8191}, \frac{4513}{8191}$$
$$\frac{1293}{8191}, \frac{1294}{8191} \simeq \frac{3671}{8191}, \frac{3672}{8191}$$
$$\frac{1295}{8191}, \frac{1296}{8191} \simeq \frac{4517}{8191}, \frac{3674}{8191}$$
$$\frac{1299}{8191}, \frac{1300}{8191} \simeq \frac{3641}{8191}, \frac{3654}{8191}$$
$$\frac{1301}{8191}, \frac{1302}{8191} \simeq \frac{3647}{8191}, \frac{3648}{8191}$$
$$\frac{1303}{8191}, \frac{1304}{8191} \simeq \frac{3645}{8191}, \frac{3650}{8191}$$
$$\frac{1305}{8191}, \frac{1314}{8191} \simeq \frac{3659}{8191}, \frac{4532}{8191}$$
$$\frac{1306}{8191}, \frac{1313}{8191} \simeq \frac{3660}{8191}, \frac{4531}{8191}$$
$$\frac{1307}{8191}, \frac{1308}{8191} \simeq \frac{4526}{8191}, \frac{4529}{8191}$$
$$\frac{1309}{8191}, \frac{1310}{8191} \simeq \frac{3655}{8191}, \frac{3656}{8191}$$
$$\frac{1311}{8191}, \frac{1312}{8191} \simeq \frac{4533}{8191}, \frac{3658}{8191}$$

$$\frac{1315}{8191},\frac{1316}{8191} \simeq \frac{4534}{8191},\frac{3657}{8191}$$
$$\frac{1317}{8191},\frac{1318}{8191} \simeq \frac{4239}{8191},\frac{4240}{8191}$$
$$\frac{1319}{8191},\frac{1320}{8191} \simeq \frac{3954}{8191},\frac{4237}{8191}$$
$$\frac{1321}{8191},\frac{1586}{8191} \simeq \frac{3972}{8191},\frac{4219}{8191}$$
$$\frac{1322}{8191},\frac{1329}{8191} \simeq \frac{4220}{8191},\frac{4227}{8191}$$
$$\frac{1323}{8191},\frac{1324}{8191} \simeq \frac{4225}{8191},\frac{4222}{8191}$$
$$\frac{1325}{8191},\frac{1326}{8191} \simeq \frac{3959}{8191},\frac{4232}{8191}$$
$$\frac{1327}{8191},\frac{1328}{8191} \simeq \frac{3962}{8191},\frac{4229}{8191}$$
$$\frac{1330}{8191},\frac{1577}{8191} \simeq \frac{3963}{8191},\frac{4228}{8191}$$
$$\frac{1331}{8191},\frac{1332}{8191} \simeq \frac{3993}{8191},\frac{4198}{8191}$$
$$\frac{1333}{8191},\frac{1334}{8191} \simeq \frac{3999}{8191},\frac{4000}{8191}$$
$$\frac{1335}{8191},\frac{1336}{8191} \simeq \frac{4002}{8191},\frac{4189}{8191}$$
$$\frac{1337}{8191},\frac{1346}{8191} \simeq \frac{3988}{8191},\frac{4203}{8191}$$
$$\frac{1338}{8191},\frac{1345}{8191} \simeq \frac{3987}{8191},\frac{4204}{8191}$$
$$\frac{1339}{8191},\frac{1340}{8191} \simeq \frac{3982}{8191},\frac{4209}{8191}$$
$$\frac{1341}{8191},\frac{1342}{8191} \simeq \frac{3992}{8191},\frac{4199}{8191}$$
$$\frac{1343}{8191},\frac{1344}{8191} \simeq \frac{4202}{8191},\frac{3989}{8191}$$
$$\frac{1347}{8191},\frac{1348}{8191} \simeq \frac{3990}{8191},\frac{4201}{8191}$$

$$\frac{1349}{8191},\frac{1574}{8191} \simeq \frac{3984}{8191},\frac{4207}{8191}$$
$$\frac{1350}{8191},\frac{1573}{8191} \simeq \frac{3983}{8191},\frac{4208}{8191}$$
$$\frac{1351}{8191},\frac{1352}{8191} \simeq \frac{4205}{8191},\frac{3986}{8191}$$
$$\frac{1353}{8191},\frac{1426}{8191} \simeq \frac{4059}{8191},\frac{4132}{8191}$$
$$\frac{1354}{8191},\frac{1425}{8191} \simeq \frac{4060}{8191},\frac{4131}{8191}$$
$$\frac{1355}{8191},\frac{1356}{8191} \simeq \frac{4062}{8191},\frac{4129}{8191}$$
$$\frac{1357}{8191},\frac{1358}{8191} \simeq \frac{4072}{8191},\frac{4119}{8191}$$
$$\frac{1359}{8191},\frac{1360}{8191} \simeq \frac{4069}{8191},\frac{4122}{8191}$$
$$\frac{1361}{8191},\frac{1418}{8191} \simeq \frac{4067}{8191},\frac{4124}{8191}$$
$$\frac{1362}{8191},\frac{1417}{8191} \simeq \frac{4068}{8191},\frac{4123}{8191}$$
$$\frac{1363}{8191},\frac{1364}{8191} \simeq \frac{4089}{8191},\frac{4102}{8191}$$
$$\frac{1365}{8191},\frac{1366}{8191} \simeq \frac{4096}{8191},\frac{4095}{8191}$$
$$\frac{1367}{8191},\frac{1368}{8191} \simeq \frac{4093}{8191},\frac{4098}{8191}$$
$$\frac{1369}{8191},\frac{1378}{8191} \simeq \frac{4084}{8191},\frac{4107}{8191}$$
$$\frac{1370}{8191},\frac{1377}{8191} \simeq \frac{4083}{8191},\frac{4108}{8191}$$
$$\frac{1371}{8191},\frac{1372}{8191} \simeq \frac{4078}{8191},\frac{4113}{8191}$$
$$\frac{1373}{8191},\frac{1374}{8191} \simeq \frac{4088}{8191},\frac{4103}{8191}$$
$$\frac{1375}{8191},\frac{1376}{8191} \simeq \frac{4085}{8191},\frac{4106}{8191}$$

$$\frac{1379}{8191}, \frac{1380}{8191} \simeq \frac{4086}{8191}, \frac{4105}{8191}$$
$$\frac{1381}{8191}, \frac{1414}{8191} \simeq \frac{4047}{8191}, \frac{4144}{8191}$$
$$\frac{1382}{8191}, \frac{1413}{8191} \simeq \frac{4048}{8191}, \frac{4143}{8191}$$
$$\frac{1383}{8191}, \frac{1384}{8191} \simeq \frac{4045}{8191}, \frac{4146}{8191}$$
$$\frac{1385}{8191}, \frac{1394}{8191} \simeq \frac{4027}{8191}, \frac{4164}{8191}$$
$$\frac{1386}{8191}, \frac{1393}{8191} \simeq \frac{4028}{8191}, \frac{4163}{8191}$$
$$\frac{1387}{8191}, \frac{1388}{8191} \simeq \frac{4033}{8191}, \frac{4158}{8191}$$
$$\frac{1389}{8191}, \frac{1390}{8191} \simeq \frac{4023}{8191}, \frac{4168}{8191}$$
$$\frac{1391}{8191}, \frac{1392}{8191} \simeq \frac{4026}{8191}, \frac{4165}{8191}$$
$$\frac{1395}{8191}, \frac{1396}{8191} \simeq \frac{4057}{8191}, \frac{4134}{8191}$$
$$\frac{1397}{8191}, \frac{1398}{8191} \simeq \frac{4063}{8191}, \frac{4128}{8191}$$
$$\frac{1399}{8191}, \frac{1400}{8191} \simeq \frac{4066}{8191}, \frac{4125}{8191}$$
$$\frac{1401}{8191}, \frac{1410}{8191} \simeq \frac{4052}{8191}, \frac{4139}{8191}$$
$$\frac{1402}{8191}, \frac{1409}{8191} \simeq \frac{4051}{8191}, \frac{4140}{8191}$$
$$\frac{1403}{8191}, \frac{1404}{8191} \simeq \frac{4046}{8191}, \frac{4145}{8191}$$
$$\frac{1405}{8191}, \frac{1406}{8191} \simeq \frac{4056}{8191}, \frac{4135}{8191}$$
$$\frac{1407}{8191}, \frac{1408}{8191} \simeq \frac{4138}{8191}, \frac{4053}{8191}$$
$$\frac{1411}{8191}, \frac{1412}{8191} \simeq \frac{4054}{8191}, \frac{4137}{8191}$$
$$\frac{1415}{8191}, \frac{1416}{8191} \simeq \frac{4050}{8191}, \frac{4141}{8191}$$
$$\frac{1419}{8191}, \frac{1420}{8191} \simeq \frac{4065}{8191}, \frac{4126}{8191}$$
$$\frac{1421}{8191}, \frac{1422}{8191} \simeq \frac{4055}{8191}, \frac{4136}{8191}$$
$$\frac{1423}{8191}, \frac{1424}{8191} \simeq \frac{4133}{8191}, \frac{4058}{8191}$$
$$\frac{1427}{8191}, \frac{1428}{8191} \simeq \frac{3910}{8191}, \frac{4281}{8191}$$
$$\frac{1429}{8191}, \frac{1430}{8191} \simeq \frac{3903}{8191}, \frac{3904}{8191}$$
$$\frac{1431}{8191}, \frac{1432}{8191} \simeq \frac{4285}{8191}, \frac{4290}{8191}$$
$$\frac{1433}{8191}, \frac{1442}{8191} \simeq \frac{3892}{8191}, \frac{4299}{8191}$$
$$\frac{1434}{8191}, \frac{1441}{8191} \simeq \frac{3891}{8191}, \frac{4300}{8191}$$
$$\frac{1435}{8191}, \frac{1436}{8191} \simeq \frac{3886}{8191}, \frac{4305}{8191}$$
$$\frac{1437}{8191}, \frac{1438}{8191} \simeq \frac{3896}{8191}, \frac{4295}{8191}$$
$$\frac{1439}{8191}, \frac{1440}{8191} \simeq \frac{4298}{8191}, \frac{3893}{8191}$$
$$\frac{1443}{8191}, \frac{1444}{8191} \simeq \frac{3894}{8191}, \frac{4297}{8191}$$
$$\frac{1445}{8191}, \frac{1478}{8191} \simeq \frac{4367}{8191}, \frac{4368}{8191}$$
$$\frac{1446}{8191}, \frac{1477}{8191} \simeq \frac{4335}{8191}, \frac{4336}{8191}$$
$$\frac{1447}{8191}, \frac{1448}{8191} \simeq \frac{4338}{8191}, \frac{4365}{8191}$$
$$\frac{1449}{8191}, \frac{1458}{8191} \simeq \frac{4347}{8191}, \frac{4356}{8191}$$
$$\frac{1450}{8191}, \frac{1457}{8191} \simeq \frac{4348}{8191}, \frac{4355}{8191}$$

$$\frac{1451}{8191},\frac{1452}{8191} \simeq \frac{4350}{8191},\frac{4353}{8191}$$
$$\frac{1453}{8191},\frac{1454}{8191} \simeq \frac{4344}{8191},\frac{4359}{8191}$$
$$\frac{1455}{8191},\frac{1456}{8191} \simeq \frac{4346}{8191},\frac{4357}{8191}$$
$$\frac{1459}{8191},\frac{1460}{8191} \simeq \frac{3814}{8191},\frac{4377}{8191}$$
$$\frac{1461}{8191},\frac{1462}{8191} \simeq \frac{4383}{8191},\frac{4384}{8191}$$
$$\frac{1463}{8191},\frac{1464}{8191} \simeq \frac{3805}{8191},\frac{4386}{8191}$$
$$\frac{1465}{8191},\frac{1474}{8191} \simeq \frac{3819}{8191},\frac{4372}{8191}$$
$$\frac{1466}{8191},\frac{1473}{8191} \simeq \frac{3820}{8191},\frac{4371}{8191}$$
$$\frac{1467}{8191},\frac{1468}{8191} \simeq \frac{4337}{8191},\frac{4366}{8191}$$
$$\frac{1469}{8191},\frac{1470}{8191} \simeq \frac{4375}{8191},\frac{4376}{8191}$$
$$\frac{1471}{8191},\frac{1472}{8191} \simeq \frac{4373}{8191},\frac{3818}{8191}$$
$$\frac{1475}{8191},\frac{1476}{8191} \simeq \frac{4374}{8191},\frac{3817}{8191}$$
$$\frac{1479}{8191},\frac{1480}{8191} \simeq \frac{4370}{8191},\frac{3821}{8191}$$
$$\frac{1481}{8191},\frac{1554}{8191} \simeq \frac{3931}{8191},\frac{4260}{8191}$$
$$\frac{1482}{8191},\frac{1553}{8191} \simeq \frac{3932}{8191},\frac{4259}{8191}$$
$$\frac{1483}{8191},\frac{1548}{8191} \simeq \frac{3934}{8191},\frac{4257}{8191}$$
$$\frac{1484}{8191},\frac{1547}{8191} \simeq \frac{3937}{8191},\frac{4254}{8191}$$
$$\frac{1485}{8191},\frac{1486}{8191} \simeq \frac{3944}{8191},\frac{4247}{8191}$$
$$\frac{1487}{8191},\frac{1488}{8191} \simeq \frac{4250}{8191},\frac{3941}{8191}$$
$$\frac{1489}{8191},\frac{1546}{8191} \simeq \frac{3939}{8191},\frac{4252}{8191}$$
$$\frac{1490}{8191},\frac{1545}{8191} \simeq \frac{3940}{8191},\frac{4251}{8191}$$
$$\frac{1491}{8191},\frac{1492}{8191} \simeq \frac{3974}{8191},\frac{4217}{8191}$$
$$\frac{1493}{8191},\frac{1494}{8191} \simeq \frac{3967}{8191},\frac{3968}{8191}$$
$$\frac{1495}{8191},\frac{1496}{8191} \simeq \frac{3965}{8191},\frac{3970}{8191}$$
$$\frac{1497}{8191},\frac{1506}{8191} \simeq \frac{3979}{8191},\frac{4212}{8191}$$
$$\frac{1498}{8191},\frac{1505}{8191} \simeq \frac{3980}{8191},\frac{4211}{8191}$$
$$\frac{1499}{8191},\frac{1500}{8191} \simeq \frac{3985}{8191},\frac{4206}{8191}$$
$$\frac{1501}{8191},\frac{1502}{8191} \simeq \frac{3975}{8191},\frac{4216}{8191}$$
$$\frac{1503}{8191},\frac{1504}{8191} \simeq \frac{4213}{8191},\frac{3978}{8191}$$
$$\frac{1507}{8191},\frac{1508}{8191} \simeq \frac{4214}{8191},\frac{3977}{8191}$$
$$\frac{1509}{8191},\frac{1542}{8191} \simeq \frac{3919}{8191},\frac{3920}{8191}$$
$$\frac{1510}{8191},\frac{1541}{8191} \simeq \frac{4271}{8191},\frac{4272}{8191}$$
$$\frac{1511}{8191},\frac{1512}{8191} \simeq \frac{4274}{8191},\frac{3917}{8191}$$
$$\frac{1513}{8191},\frac{1522}{8191} \simeq \frac{3899}{8191},\frac{4292}{8191}$$
$$\frac{1514}{8191},\frac{1521}{8191} \simeq \frac{3900}{8191},\frac{4291}{8191}$$
$$\frac{1515}{8191},\frac{1516}{8191} \simeq \frac{4286}{8191},\frac{4289}{8191}$$

$$\frac{1517}{8191}, \frac{1518}{8191} \simeq \frac{3895}{8191}, \frac{4296}{8191}$$
$$\frac{1519}{8191}, \frac{1520}{8191} \simeq \frac{4293}{8191}, \frac{3898}{8191}$$
$$\frac{1523}{8191}, \frac{1524}{8191} \simeq \frac{4262}{8191}, \frac{3929}{8191}$$
$$\frac{1525}{8191}, \frac{1526}{8191} \simeq \frac{3935}{8191}, \frac{3936}{8191}$$
$$\frac{1527}{8191}, \frac{1528}{8191} \simeq \frac{4253}{8191}, \frac{3938}{8191}$$
$$\frac{1529}{8191}, \frac{1538}{8191} \simeq \frac{4267}{8191}, \frac{3924}{8191}$$
$$\frac{1530}{8191}, \frac{1537}{8191} \simeq \frac{4268}{8191}, \frac{3923}{8191}$$
$$\frac{1531}{8191}, \frac{1532}{8191} \simeq \frac{4273}{8191}, \frac{3918}{8191}$$
$$\frac{1533}{8191}, \frac{1534}{8191} \simeq \frac{4263}{8191}, \frac{3928}{8191}$$
$$\frac{1535}{8191}, \frac{1536}{8191} \simeq \frac{4266}{8191}, \frac{3925}{8191}$$
$$\frac{1539}{8191}, \frac{1540}{8191} \simeq \frac{4265}{8191}, \frac{3926}{8191}$$
$$\frac{1543}{8191}, \frac{1544}{8191} \simeq \frac{4269}{8191}, \frac{3922}{8191}$$
$$\frac{1549}{8191}, \frac{1550}{8191} \simeq \frac{3927}{8191}, \frac{4264}{8191}$$
$$\frac{1551}{8191}, \frac{1552}{8191} \simeq \frac{4261}{8191}, \frac{3930}{8191}$$
$$\frac{1555}{8191}, \frac{1556}{8191} \simeq \frac{3897}{8191}, \frac{4294}{8191}$$
$$\frac{1557}{8191}, \frac{1558}{8191} \simeq \frac{4287}{8191}, \frac{4288}{8191}$$
$$\frac{1559}{8191}, \frac{1560}{8191} \simeq \frac{3901}{8191}, \frac{3906}{8191}$$
$$\frac{1561}{8191}, \frac{1570}{8191} \simeq \frac{3915}{8191}, \frac{4276}{8191}$$

$$\frac{1562}{8191}, \frac{1569}{8191} \simeq \frac{3916}{8191}, \frac{4275}{8191}$$
$$\frac{1563}{8191}, \frac{1564}{8191} \simeq \frac{3921}{8191}, \frac{4270}{8191}$$
$$\frac{1565}{8191}, \frac{1566}{8191} \simeq \frac{3911}{8191}, \frac{4280}{8191}$$
$$\frac{1567}{8191}, \frac{1568}{8191} \simeq \frac{4277}{8191}, \frac{3914}{8191}$$
$$\frac{1571}{8191}, \frac{1572}{8191} \simeq \frac{4278}{8191}, \frac{3913}{8191}$$
$$\frac{1575}{8191}, \frac{1576}{8191} \simeq \frac{4210}{8191}, \frac{3981}{8191}$$
$$\frac{1578}{8191}, \frac{1585}{8191} \simeq \frac{3964}{8191}, \frac{3971}{8191}$$
$$\frac{1579}{8191}, \frac{1580}{8191} \simeq \frac{3966}{8191}, \frac{3969}{8191}$$
$$\frac{1581}{8191}, \frac{1582}{8191} \simeq \frac{3976}{8191}, \frac{4215}{8191}$$
$$\frac{1583}{8191}, \frac{1584}{8191} \simeq \frac{4218}{8191}, \frac{3973}{8191}$$
$$\frac{1587}{8191}, \frac{1588}{8191} \simeq \frac{3942}{8191}, \frac{4249}{8191}$$
$$\frac{1589}{8191}, \frac{1590}{8191} \simeq \frac{4255}{8191}, \frac{4256}{8191}$$
$$\frac{1591}{8191}, \frac{1592}{8191} \simeq \frac{4258}{8191}, \frac{3933}{8191}$$
$$\frac{1593}{8191}, \frac{1602}{8191} \simeq \frac{3947}{8191}, \frac{4244}{8191}$$
$$\frac{1594}{8191}, \frac{1601}{8191} \simeq \frac{3948}{8191}, \frac{4243}{8191}$$
$$\frac{1595}{8191}, \frac{1596}{8191} \simeq \frac{4238}{8191}, \frac{3953}{8191}$$
$$\frac{1597}{8191}, \frac{1598}{8191} \simeq \frac{4248}{8191}, \frac{3943}{8191}$$
$$\frac{1599}{8191}, \frac{1600}{8191} \simeq \frac{4245}{8191}, \frac{3946}{8191}$$

$$\frac{1603}{8191}, \frac{1604}{8191} \simeq \frac{4246}{8191}, \frac{3945}{8191}$$
$$\frac{1605}{8191}, \frac{1606}{8191} \simeq \frac{3951}{8191}, \frac{3952}{8191}$$
$$\frac{1607}{8191}, \frac{1608}{8191} \simeq \frac{4242}{8191}, \frac{3949}{8191}$$
$$\frac{1609}{8191}, \frac{2194}{8191} \simeq \frac{4827}{8191}, \frac{4828}{8191}$$
$$\frac{1610}{8191}, \frac{2193}{8191} \simeq \frac{4835}{8191}, \frac{4836}{8191}$$
$$\frac{1611}{8191}, \frac{1612}{8191} \simeq \frac{4830}{8191}, \frac{4833}{8191}$$
$$\frac{1613}{8191}, \frac{1614}{8191} \simeq \frac{4887}{8191}, \frac{4888}{8191}$$
$$\frac{1615}{8191}, \frac{1616}{8191} \simeq \frac{4901}{8191}, \frac{4890}{8191}$$
$$\frac{1617}{8191}, \frac{2186}{8191} \simeq \frac{4899}{8191}, \frac{4900}{8191}$$
$$\frac{1618}{8191}, \frac{2185}{8191} \simeq \frac{4891}{8191}, \frac{4892}{8191}$$
$$\frac{1619}{8191}, \frac{1620}{8191} \simeq \frac{4857}{8191}, \frac{4870}{8191}$$
$$\frac{1621}{8191}, \frac{1622}{8191} \simeq \frac{4864}{8191}, \frac{4863}{8191}$$
$$\frac{1623}{8191}, \frac{1624}{8191} \simeq \frac{4861}{8191}, \frac{4866}{8191}$$
$$\frac{1625}{8191}, \frac{1634}{8191} \simeq \frac{4875}{8191}, \frac{4876}{8191}$$
$$\frac{1626}{8191}, \frac{1633}{8191} \simeq \frac{4851}{8191}, \frac{4852}{8191}$$
$$\frac{1627}{8191}, \frac{1628}{8191} \simeq \frac{4846}{8191}, \frac{4849}{8191}$$
$$\frac{1629}{8191}, \frac{1630}{8191} \simeq \frac{4871}{8191}, \frac{4872}{8191}$$
$$\frac{1631}{8191}, \frac{1632}{8191} \simeq \frac{4874}{8191}, \frac{4853}{8191}$$
$$\frac{1635}{8191}, \frac{1636}{8191} \simeq \frac{4854}{8191}, \frac{4873}{8191}$$
$$\frac{1637}{8191}, \frac{1638}{8191} \simeq \frac{4911}{8191}, \frac{4912}{8191}$$
$$\frac{1639}{8191}, \frac{1640}{8191} \simeq \frac{3282}{8191}, \frac{3277}{8191}$$
$$\frac{1641}{8191}, \frac{1650}{8191} \simeq \frac{3259}{8191}, \frac{3260}{8191}$$
$$\frac{1642}{8191}, \frac{1649}{8191} \simeq \frac{3267}{8191}, \frac{3268}{8191}$$
$$\frac{1643}{8191}, \frac{1644}{8191} \simeq \frac{3262}{8191}, \frac{3265}{8191}$$
$$\frac{1645}{8191}, \frac{1646}{8191} \simeq \frac{3255}{8191}, \frac{3256}{8191}$$
$$\frac{1647}{8191}, \frac{1648}{8191} \simeq \frac{3269}{8191}, \frac{3258}{8191}$$
$$\frac{1651}{8191}, \frac{1668}{8191} \simeq \frac{3366}{8191}, \frac{3289}{8191}$$
$$\frac{1652}{8191}, \frac{1667}{8191} \simeq \frac{3369}{8191}, \frac{3286}{8191}$$
$$\frac{1653}{8191}, \frac{1654}{8191} \simeq \frac{3359}{8191}, \frac{3360}{8191}$$
$$\frac{1655}{8191}, \frac{1656}{8191} \simeq \frac{3362}{8191}, \frac{3357}{8191}$$
$$\frac{1657}{8191}, \frac{1666}{8191} \simeq \frac{3372}{8191}, \frac{3371}{8191}$$
$$\frac{1658}{8191}, \frac{1665}{8191} \simeq \frac{3284}{8191}, \frac{3283}{8191}$$
$$\frac{1659}{8191}, \frac{1660}{8191} \simeq \frac{3281}{8191}, \frac{3278}{8191}$$
$$\frac{1661}{8191}, \frac{1662}{8191} \simeq \frac{3368}{8191}, \frac{3367}{8191}$$
$$\frac{1663}{8191}, \frac{1664}{8191} \simeq \frac{3370}{8191}, \frac{3285}{8191}$$
$$\frac{1669}{8191}, \frac{1670}{8191} \simeq \frac{3279}{8191}, \frac{3280}{8191}$$

$$\frac{1671}{8191},\frac{1672}{8191} \simeq \frac{3378}{8191},\frac{3373}{8191}$$
$$\frac{1673}{8191},\frac{2130}{8191} \simeq \frac{3355}{8191},\frac{3356}{8191}$$
$$\frac{1674}{8191},\frac{2129}{8191} \simeq \frac{3363}{8191},\frac{3364}{8191}$$
$$\frac{1675}{8191},\frac{1676}{8191} \simeq \frac{3358}{8191},\frac{3361}{8191}$$
$$\frac{1677}{8191},\frac{1678}{8191} \simeq \frac{3287}{8191},\frac{3288}{8191}$$
$$\frac{1679}{8191},\frac{1680}{8191} \simeq \frac{3301}{8191},\frac{3290}{8191}$$
$$\frac{1681}{8191},\frac{2122}{8191} \simeq \frac{3299}{8191},\frac{3300}{8191}$$
$$\frac{1682}{8191},\frac{2121}{8191} \simeq \frac{3291}{8191},\frac{3292}{8191}$$
$$\frac{1683}{8191},\frac{1828}{8191} \simeq \frac{3001}{8191},\frac{2998}{8191}$$
$$\frac{1684}{8191},\frac{1827}{8191} \simeq \frac{3014}{8191},\frac{3017}{8191}$$
$$\frac{1685}{8191},\frac{1686}{8191} \simeq \frac{3008}{8191},\frac{3007}{8191}$$
$$\frac{1687}{8191},\frac{1688}{8191} \simeq \frac{3005}{8191},\frac{3010}{8191}$$
$$\frac{1689}{8191},\frac{1826}{8191} \simeq \frac{3019}{8191},\frac{3116}{8191}$$
$$\frac{1690}{8191},\frac{1825}{8191} \simeq \frac{2996}{8191},\frac{2995}{8191}$$
$$\frac{1691}{8191},\frac{1692}{8191} \simeq \frac{2993}{8191},\frac{3118}{8191}$$
$$\frac{1693}{8191},\frac{1694}{8191} \simeq \frac{3143}{8191},\frac{3144}{8191}$$
$$\frac{1695}{8191},\frac{1696}{8191} \simeq \frac{3146}{8191},\frac{3125}{8191}$$
$$\frac{1697}{8191},\frac{1818}{8191} \simeq \frac{3123}{8191},\frac{3124}{8191}$$
$$\frac{1698}{8191},\frac{1817}{8191} \simeq \frac{2988}{8191},\frac{3147}{8191}$$
$$\frac{1699}{8191},\frac{1812}{8191} \simeq \frac{3142}{8191},\frac{3145}{8191}$$
$$\frac{1700}{8191},\frac{1811}{8191} \simeq \frac{3126}{8191},\frac{3129}{8191}$$
$$\frac{1701}{8191},\frac{1702}{8191} \simeq \frac{3055}{8191},\frac{3056}{8191}$$
$$\frac{1703}{8191},\frac{1704}{8191} \simeq \frac{3085}{8191},\frac{3090}{8191}$$
$$\frac{1705}{8191},\frac{1714}{8191} \simeq \frac{3067}{8191},\frac{3068}{8191}$$
$$\frac{1706}{8191},\frac{1713}{8191} \simeq \frac{3075}{8191},\frac{3076}{8191}$$
$$\frac{1707}{8191},\frac{1708}{8191} \simeq \frac{3070}{8191},\frac{3073}{8191}$$
$$\frac{1709}{8191},\frac{1710}{8191} \simeq \frac{3063}{8191},\frac{3064}{8191}$$
$$\frac{1711}{8191},\frac{1712}{8191} \simeq \frac{3066}{8191},\frac{3077}{8191}$$
$$\frac{1715}{8191},\frac{1732}{8191} \simeq \frac{3094}{8191},\frac{3033}{8191}$$
$$\frac{1716}{8191},\frac{1731}{8191} \simeq \frac{3046}{8191},\frac{3049}{8191}$$
$$\frac{1717}{8191},\frac{1718}{8191} \simeq \frac{3039}{8191},\frac{3040}{8191}$$
$$\frac{1719}{8191},\frac{1720}{8191} \simeq \frac{3037}{8191},\frac{3042}{8191}$$
$$\frac{1721}{8191},\frac{1730}{8191} \simeq \frac{3051}{8191},\frac{3084}{8191}$$
$$\frac{1722}{8191},\frac{1729}{8191} \simeq \frac{3091}{8191},\frac{3092}{8191}$$
$$\frac{1723}{8191},\frac{1724}{8191} \simeq \frac{3086}{8191},\frac{3089}{8191}$$
$$\frac{1725}{8191},\frac{1726}{8191} \simeq \frac{3047}{8191},\frac{3048}{8191}$$

$$\frac{1727}{8191},\frac{1728}{8191} \simeq \frac{3093}{8191},\frac{3050}{8191}$$
$$\frac{1733}{8191},\frac{1734}{8191} \simeq \frac{3087}{8191},\frac{3088}{8191}$$
$$\frac{1735}{8191},\frac{1736}{8191} \simeq \frac{3058}{8191},\frac{3053}{8191}$$
$$\frac{1737}{8191},\frac{1810}{8191} \simeq \frac{3163}{8191},\frac{3164}{8191}$$
$$\frac{1738}{8191},\frac{1809}{8191} \simeq \frac{2916}{8191},\frac{2915}{8191}$$
$$\frac{1739}{8191},\frac{1740}{8191} \simeq \frac{2913}{8191},\frac{2910}{8191}$$
$$\frac{1741}{8191},\frac{1806}{8191} \simeq \frac{3159}{8191},\frac{3160}{8191}$$
$$\frac{1742}{8191},\frac{1805}{8191} \simeq \frac{2967}{8191},\frac{2968}{8191}$$
$$\frac{1743}{8191},\frac{1744}{8191} \simeq \frac{2981}{8191},\frac{2970}{8191}$$
$$\frac{1745}{8191},\frac{1802}{8191} \simeq \frac{2979}{8191},\frac{2980}{8191}$$
$$\frac{1746}{8191},\frac{1801}{8191} \simeq \frac{2971}{8191},\frac{2972}{8191}$$
$$\frac{1747}{8191},\frac{1764}{8191} \simeq \frac{2937}{8191},\frac{2934}{8191}$$
$$\frac{1748}{8191},\frac{1763}{8191} \simeq \frac{2950}{8191},\frac{2953}{8191}$$
$$\frac{1749}{8191},\frac{1750}{8191} \simeq \frac{2943}{8191},\frac{2944}{8191}$$
$$\frac{1751}{8191},\frac{1752}{8191} \simeq \frac{2941}{8191},\frac{2946}{8191}$$
$$\frac{1753}{8191},\frac{1762}{8191} \simeq \frac{2955}{8191},\frac{2924}{8191}$$
$$\frac{1754}{8191},\frac{1761}{8191} \simeq \frac{2931}{8191},\frac{2932}{8191}$$
$$\frac{1755}{8191},\frac{1756}{8191} \simeq \frac{2929}{8191},\frac{2926}{8191}$$

$$\frac{1757}{8191},\frac{1758}{8191} \simeq \frac{2951}{8191},\frac{2952}{8191}$$
$$\frac{1759}{8191},\frac{1760}{8191} \simeq \frac{2954}{8191},\frac{2933}{8191}$$
$$\frac{1765}{8191},\frac{1766}{8191} \simeq \frac{3119}{8191},\frac{3120}{8191}$$
$$\frac{1767}{8191},\frac{1768}{8191} \simeq \frac{3154}{8191},\frac{3149}{8191}$$
$$\frac{1769}{8191},\frac{1778}{8191} \simeq \frac{3131}{8191},\frac{3132}{8191}$$
$$\frac{1770}{8191},\frac{1777}{8191} \simeq \frac{3139}{8191},\frac{3140}{8191}$$
$$\frac{1771}{8191},\frac{1772}{8191} \simeq \frac{3134}{8191},\frac{3137}{8191}$$
$$\frac{1773}{8191},\frac{1774}{8191} \simeq \frac{3127}{8191},\frac{3128}{8191}$$
$$\frac{1775}{8191},\frac{1776}{8191} \simeq \frac{3141}{8191},\frac{3130}{8191}$$
$$\frac{1779}{8191},\frac{1796}{8191} \simeq \frac{3158}{8191},\frac{2969}{8191}$$
$$\frac{1780}{8191},\frac{1795}{8191} \simeq \frac{2985}{8191},\frac{2982}{8191}$$
$$\frac{1781}{8191},\frac{1782}{8191} \simeq \frac{2975}{8191},\frac{2976}{8191}$$
$$\frac{1783}{8191},\frac{1784}{8191} \simeq \frac{2978}{8191},\frac{2973}{8191}$$
$$\frac{1785}{8191},\frac{1794}{8191} \simeq \frac{3148}{8191},\frac{2987}{8191}$$
$$\frac{1786}{8191},\frac{1793}{8191} \simeq \frac{3156}{8191},\frac{3155}{8191}$$
$$\frac{1787}{8191},\frac{1788}{8191} \simeq \frac{3153}{8191},\frac{3150}{8191}$$
$$\frac{1789}{8191},\frac{1790}{8191} \simeq \frac{2984}{8191},\frac{2983}{8191}$$
$$\frac{1791}{8191},\frac{1792}{8191} \simeq \frac{3157}{8191},\frac{2986}{8191}$$

$$\frac{1797}{8191}, \frac{1798}{8191} \simeq \frac{3151}{8191}, \frac{3152}{8191}$$
$$\frac{1799}{8191}, \frac{1800}{8191} \simeq \frac{3122}{8191}, \frac{2989}{8191}$$
$$\frac{1803}{8191}, \frac{1804}{8191} \simeq \frac{2974}{8191}, \frac{2977}{8191}$$
$$\frac{1807}{8191}, \frac{1808}{8191} \simeq \frac{3162}{8191}, \frac{2917}{8191}$$
$$\frac{1813}{8191}, \frac{1814}{8191} \simeq \frac{3135}{8191}, \frac{3136}{8191}$$
$$\frac{1815}{8191}, \frac{1816}{8191} \simeq \frac{3133}{8191}, \frac{3138}{8191}$$
$$\frac{1819}{8191}, \frac{1820}{8191} \simeq \frac{3121}{8191}, \frac{2990}{8191}$$
$$\frac{1821}{8191}, \frac{1822}{8191} \simeq \frac{3015}{8191}, \frac{3016}{8191}$$
$$\frac{1823}{8191}, \frac{1824}{8191} \simeq \frac{3018}{8191}, \frac{2997}{8191}$$
$$\frac{1829}{8191}, \frac{1830}{8191} \simeq \frac{3439}{8191}, \frac{3440}{8191}$$
$$\frac{1831}{8191}, \frac{1832}{8191} \simeq \frac{3474}{8191}, \frac{3437}{8191}$$
$$\frac{1833}{8191}, \frac{1842}{8191} \simeq \frac{3451}{8191}, \frac{3452}{8191}$$
$$\frac{1834}{8191}, \frac{1841}{8191} \simeq \frac{3459}{8191}, \frac{3460}{8191}$$
$$\frac{1835}{8191}, \frac{1836}{8191} \simeq \frac{3454}{8191}, \frac{3457}{8191}$$
$$\frac{1837}{8191}, \frac{1838}{8191} \simeq \frac{3447}{8191}, \frac{3448}{8191}$$
$$\frac{1839}{8191}, \frac{1840}{8191} \simeq \frac{3461}{8191}, \frac{3450}{8191}$$
$$\frac{1843}{8191}, \frac{2116}{8191} \simeq \frac{4710}{8191}, \frac{3481}{8191}$$
$$\frac{1844}{8191}, \frac{2115}{8191} \simeq \frac{4713}{8191}, \frac{3478}{8191}$$

$$\frac{1845}{8191}, \frac{1846}{8191} \simeq \frac{4703}{8191}, \frac{4704}{8191}$$
$$\frac{1847}{8191}, \frac{1848}{8191} \simeq \frac{4706}{8191}, \frac{4701}{8191}$$
$$\frac{1849}{8191}, \frac{2114}{8191} \simeq \frac{4716}{8191}, \frac{4715}{8191}$$
$$\frac{1850}{8191}, \frac{1857}{8191} \simeq \frac{4755}{8191}, \frac{4756}{8191}$$
$$\frac{1851}{8191}, \frac{1852}{8191} \simeq \frac{4753}{8191}, \frac{4750}{8191}$$
$$\frac{1853}{8191}, \frac{1854}{8191} \simeq \frac{3432}{8191}, \frac{3431}{8191}$$
$$\frac{1855}{8191}, \frac{1856}{8191} \simeq \frac{4757}{8191}, \frac{3434}{8191}$$
$$\frac{1858}{8191}, \frac{2105}{8191} \simeq \frac{3436}{8191}, \frac{3435}{8191}$$
$$\frac{1859}{8191}, \frac{2100}{8191} \simeq \frac{4758}{8191}, \frac{3433}{8191}$$
$$\frac{1860}{8191}, \frac{2099}{8191} \simeq \frac{4761}{8191}, \frac{3430}{8191}$$
$$\frac{1861}{8191}, \frac{1862}{8191} \simeq \frac{4751}{8191}, \frac{4752}{8191}$$
$$\frac{1863}{8191}, \frac{2088}{8191} \simeq \frac{4754}{8191}, \frac{4717}{8191}$$
$$\frac{1864}{8191}, \frac{2087}{8191} \simeq \frac{4749}{8191}, \frac{4722}{8191}$$
$$\frac{1865}{8191}, \frac{1938}{8191} \simeq \frac{4571}{8191}, \frac{4644}{8191}$$
$$\frac{1866}{8191}, \frac{1937}{8191} \simeq \frac{4572}{8191}, \frac{4579}{8191}$$
$$\frac{1867}{8191}, \frac{1868}{8191} \simeq \frac{4574}{8191}, \frac{4577}{8191}$$
$$\frac{1869}{8191}, \frac{1870}{8191} \simeq \frac{4631}{8191}, \frac{4632}{8191}$$
$$\frac{1871}{8191}, \frac{1872}{8191} \simeq \frac{4645}{8191}, \frac{4570}{8191}$$

$$\frac{1873}{8191}, \frac{1930}{8191} \simeq \frac{4636}{8191}, \frac{4643}{8191}$$
$$\frac{1874}{8191}, \frac{1929}{8191} \simeq \frac{4580}{8191}, \frac{4635}{8191}$$
$$\frac{1875}{8191}, \frac{1892}{8191} \simeq \frac{4601}{8191}, \frac{4614}{8191}$$
$$\frac{1876}{8191}, \frac{1891}{8191} \simeq \frac{4598}{8191}, \frac{4617}{8191}$$
$$\frac{1877}{8191}, \frac{1878}{8191} \simeq \frac{4607}{8191}, \frac{4608}{8191}$$
$$\frac{1879}{8191}, \frac{1880}{8191} \simeq \frac{4605}{8191}, \frac{4610}{8191}$$
$$\frac{1881}{8191}, \frac{1890}{8191} \simeq \frac{4596}{8191}, \frac{4619}{8191}$$
$$\frac{1882}{8191}, \frac{1889}{8191} \simeq \frac{4595}{8191}, \frac{4620}{8191}$$
$$\frac{1883}{8191}, \frac{1884}{8191} \simeq \frac{4590}{8191}, \frac{4593}{8191}$$
$$\frac{1885}{8191}, \frac{1886}{8191} \simeq \frac{4615}{8191}, \frac{4616}{8191}$$
$$\frac{1887}{8191}, \frac{1888}{8191} \simeq \frac{4618}{8191}, \frac{4597}{8191}$$
$$\frac{1893}{8191}, \frac{1894}{8191} \simeq \frac{4655}{8191}, \frac{4656}{8191}$$
$$\frac{1895}{8191}, \frac{1928}{8191} \simeq \frac{4658}{8191}, \frac{4557}{8191}$$
$$\frac{1896}{8191}, \frac{1927}{8191} \simeq \frac{4653}{8191}, \frac{4562}{8191}$$
$$\frac{1897}{8191}, \frac{1906}{8191} \simeq \frac{4539}{8191}, \frac{4548}{8191}$$
$$\frac{1898}{8191}, \frac{1905}{8191} \simeq \frac{4540}{8191}, \frac{4547}{8191}$$
$$\frac{1899}{8191}, \frac{1900}{8191} \simeq \frac{4542}{8191}, \frac{4545}{8191}$$
$$\frac{1901}{8191}, \frac{1902}{8191} \simeq \frac{4535}{8191}, \frac{4536}{8191}$$
$$\frac{1903}{8191}, \frac{1904}{8191} \simeq \frac{4549}{8191}, \frac{4538}{8191}$$
$$\frac{1907}{8191}, \frac{1924}{8191} \simeq \frac{4646}{8191}, \frac{4569}{8191}$$
$$\frac{1908}{8191}, \frac{1923}{8191} \simeq \frac{4649}{8191}, \frac{4566}{8191}$$
$$\frac{1909}{8191}, \frac{1910}{8191} \simeq \frac{4639}{8191}, \frac{4640}{8191}$$
$$\frac{1911}{8191}, \frac{1912}{8191} \simeq \frac{4642}{8191}, \frac{4637}{8191}$$
$$\frac{1913}{8191}, \frac{1922}{8191} \simeq \frac{4651}{8191}, \frac{4564}{8191}$$
$$\frac{1914}{8191}, \frac{1921}{8191} \simeq \frac{4652}{8191}, \frac{4563}{8191}$$
$$\frac{1915}{8191}, \frac{1916}{8191} \simeq \frac{4561}{8191}, \frac{4558}{8191}$$
$$\frac{1917}{8191}, \frac{1918}{8191} \simeq \frac{4648}{8191}, \frac{4647}{8191}$$
$$\frac{1919}{8191}, \frac{1920}{8191} \simeq \frac{4650}{8191}, \frac{4565}{8191}$$
$$\frac{1925}{8191}, \frac{1926}{8191} \simeq \frac{4559}{8191}, \frac{4560}{8191}$$
$$\frac{1931}{8191}, \frac{1932}{8191} \simeq \frac{4638}{8191}, \frac{4641}{8191}$$
$$\frac{1933}{8191}, \frac{1934}{8191} \simeq \frac{4567}{8191}, \frac{4568}{8191}$$
$$\frac{1935}{8191}, \frac{1936}{8191} \simeq \frac{4634}{8191}, \frac{4581}{8191}$$
$$\frac{1939}{8191}, \frac{2084}{8191} \simeq \frac{4806}{8191}, \frac{4793}{8191}$$
$$\frac{1940}{8191}, \frac{2083}{8191} \simeq \frac{4809}{8191}, \frac{4790}{8191}$$
$$\frac{1941}{8191}, \frac{1942}{8191} \simeq \frac{4799}{8191}, \frac{4800}{8191}$$
$$\frac{1943}{8191}, \frac{1944}{8191} \simeq \frac{4802}{8191}, \frac{4797}{8191}$$

$$\frac{1945}{8191},\frac{2082}{8191} \simeq \frac{4812}{8191},\frac{4811}{8191}$$
$$\frac{1946}{8191},\frac{2081}{8191} \simeq \frac{3380}{8191},\frac{3379}{8191}$$
$$\frac{1947}{8191},\frac{1948}{8191} \simeq \frac{3377}{8191},\frac{3374}{8191}$$
$$\frac{1949}{8191},\frac{1950}{8191} \simeq \frac{3400}{8191},\frac{3399}{8191}$$
$$\frac{1951}{8191},\frac{1952}{8191} \simeq \frac{4810}{8191},\frac{4789}{8191}$$
$$\frac{1953}{8191},\frac{2074}{8191} \simeq \frac{4788}{8191},\frac{4787}{8191}$$
$$\frac{1954}{8191},\frac{2073}{8191} \simeq \frac{3404}{8191},\frac{3403}{8191}$$
$$\frac{1955}{8191},\frac{2068}{8191} \simeq \frac{3401}{8191},\frac{3382}{8191}$$
$$\frac{1956}{8191},\frac{2067}{8191} \simeq \frac{3398}{8191},\frac{3385}{8191}$$
$$\frac{1957}{8191},\frac{1958}{8191} \simeq \frac{3311}{8191},\frac{3312}{8191}$$
$$\frac{1959}{8191},\frac{1992}{8191} \simeq \frac{3346}{8191},\frac{3341}{8191}$$
$$\frac{1960}{8191},\frac{1991}{8191} \simeq \frac{3314}{8191},\frac{3309}{8191}$$
$$\frac{1961}{8191},\frac{1970}{8191} \simeq \frac{3323}{8191},\frac{3324}{8191}$$
$$\frac{1962}{8191},\frac{1969}{8191} \simeq \frac{3331}{8191},\frac{3332}{8191}$$
$$\frac{1963}{8191},\frac{1964}{8191} \simeq \frac{3326}{8191},\frac{3329}{8191}$$
$$\frac{1965}{8191},\frac{1966}{8191} \simeq \frac{3319}{8191},\frac{3320}{8191}$$
$$\frac{1967}{8191},\frac{1968}{8191} \simeq \frac{3333}{8191},\frac{3322}{8191}$$
$$\frac{1971}{8191},\frac{1988}{8191} \simeq \frac{3353}{8191},\frac{3302}{8191}$$
$$\frac{1972}{8191},\frac{1987}{8191} \simeq \frac{3350}{8191},\frac{3305}{8191}$$
$$\frac{1973}{8191},\frac{1974}{8191} \simeq \frac{3295}{8191},\frac{3296}{8191}$$
$$\frac{1975}{8191},\frac{1976}{8191} \simeq \frac{3298}{8191},\frac{3293}{8191}$$
$$\frac{1977}{8191},\frac{1986}{8191} \simeq \frac{3308}{8191},\frac{3307}{8191}$$
$$\frac{1978}{8191},\frac{1985}{8191} \simeq \frac{3348}{8191},\frac{3347}{8191}$$
$$\frac{1979}{8191},\frac{1980}{8191} \simeq \frac{3345}{8191},\frac{3342}{8191}$$
$$\frac{1981}{8191},\frac{1982}{8191} \simeq \frac{3304}{8191},\frac{3303}{8191}$$
$$\frac{1983}{8191},\frac{1984}{8191} \simeq \frac{3349}{8191},\frac{3306}{8191}$$
$$\frac{1989}{8191},\frac{1990}{8191} \simeq \frac{3343}{8191},\frac{3344}{8191}$$
$$\frac{1993}{8191},\frac{2066}{8191} \simeq \frac{3420}{8191},\frac{3419}{8191}$$
$$\frac{1994}{8191},\frac{2065}{8191} \simeq \frac{3428}{8191},\frac{3427}{8191}$$
$$\frac{1995}{8191},\frac{1996}{8191} \simeq \frac{3425}{8191},\frac{3422}{8191}$$
$$\frac{1997}{8191},\frac{1998}{8191} \simeq \frac{4760}{8191},\frac{4759}{8191}$$
$$\frac{1999}{8191},\frac{2064}{8191} \simeq \frac{4773}{8191},\frac{3418}{8191}$$
$$\frac{2000}{8191},\frac{2063}{8191} \simeq \frac{4762}{8191},\frac{3429}{8191}$$
$$\frac{2001}{8191},\frac{2058}{8191} \simeq \frac{4772}{8191},\frac{4771}{8191}$$
$$\frac{2002}{8191},\frac{2057}{8191} \simeq \frac{4764}{8191},\frac{4763}{8191}$$
$$\frac{2003}{8191},\frac{2020}{8191} \simeq \frac{4742}{8191},\frac{4729}{8191}$$

$$\frac{2004}{8191}, \frac{2019}{8191} \simeq \frac{4745}{8191}, \frac{4726}{8191}$$
$$\frac{2005}{8191}, \frac{2006}{8191} \simeq \frac{4735}{8191}, \frac{4736}{8191}$$
$$\frac{2007}{8191}, \frac{2008}{8191} \simeq \frac{4738}{8191}, \frac{4733}{8191}$$
$$\frac{2009}{8191}, \frac{2018}{8191} \simeq \frac{4748}{8191}, \frac{4747}{8191}$$
$$\frac{2010}{8191}, \frac{2017}{8191} \simeq \frac{4724}{8191}, \frac{4723}{8191}$$
$$\frac{2011}{8191}, \frac{2012}{8191} \simeq \frac{4721}{8191}, \frac{4718}{8191}$$
$$\frac{2013}{8191}, \frac{2014}{8191} \simeq \frac{4744}{8191}, \frac{4743}{8191}$$
$$\frac{2015}{8191}, \frac{2016}{8191} \simeq \frac{4746}{8191}, \frac{4725}{8191}$$
$$\frac{2021}{8191}, \frac{2022}{8191} \simeq \frac{4784}{8191}, \frac{4783}{8191}$$
$$\frac{2023}{8191}, \frac{2056}{8191} \simeq \frac{3410}{8191}, \frac{3405}{8191}$$
$$\frac{2024}{8191}, \frac{2055}{8191} \simeq \frac{4786}{8191}, \frac{4781}{8191}$$
$$\frac{2025}{8191}, \frac{2034}{8191} \simeq \frac{3388}{8191}, \frac{3387}{8191}$$
$$\frac{2026}{8191}, \frac{2033}{8191} \simeq \frac{3396}{8191}, \frac{3395}{8191}$$
$$\frac{2027}{8191}, \frac{2028}{8191} \simeq \frac{3393}{8191}, \frac{3390}{8191}$$
$$\frac{2029}{8191}, \frac{2030}{8191} \simeq \frac{3384}{8191}, \frac{3383}{8191}$$
$$\frac{2031}{8191}, \frac{2032}{8191} \simeq \frac{3397}{8191}, \frac{3386}{8191}$$
$$\frac{2035}{8191}, \frac{2052}{8191} \simeq \frac{4774}{8191}, \frac{3417}{8191}$$
$$\frac{2036}{8191}, \frac{2051}{8191} \simeq \frac{4777}{8191}, \frac{3414}{8191}$$
$$\frac{2037}{8191}, \frac{2038}{8191} \simeq \frac{4768}{8191}, \frac{4767}{8191}$$
$$\frac{2039}{8191}, \frac{2040}{8191} \simeq \frac{4770}{8191}, \frac{4765}{8191}$$
$$\frac{2041}{8191}, \frac{2050}{8191} \simeq \frac{4780}{8191}, \frac{4779}{8191}$$
$$\frac{2042}{8191}, \frac{2049}{8191} \simeq \frac{3412}{8191}, \frac{3411}{8191}$$
$$\frac{2043}{8191}, \frac{2044}{8191} \simeq \frac{3409}{8191}, \frac{3406}{8191}$$
$$\frac{2045}{8191}, \frac{2046}{8191} \simeq \frac{4776}{8191}, \frac{4775}{8191}$$
$$\frac{2047}{8191}, \frac{2048}{8191} \simeq \frac{4778}{8191}, \frac{3413}{8191}$$
$$\frac{2053}{8191}, \frac{2054}{8191} \simeq \frac{3408}{8191}, \frac{3407}{8191}$$
$$\frac{2059}{8191}, \frac{2060}{8191} \simeq \frac{4769}{8191}, \frac{4766}{8191}$$
$$\frac{2061}{8191}, \frac{2062}{8191} \simeq \frac{3416}{8191}, \frac{3415}{8191}$$
$$\frac{2069}{8191}, \frac{2070}{8191} \simeq \frac{3391}{8191}, \frac{3392}{8191}$$
$$\frac{2071}{8191}, \frac{2072}{8191} \simeq \frac{3394}{8191}, \frac{3389}{8191}$$
$$\frac{2075}{8191}, \frac{2076}{8191} \simeq \frac{4785}{8191}, \frac{4782}{8191}$$
$$\frac{2077}{8191}, \frac{2078}{8191} \simeq \frac{4808}{8191}, \frac{4807}{8191}$$
$$\frac{2079}{8191}, \frac{2080}{8191} \simeq \frac{3402}{8191}, \frac{3381}{8191}$$
$$\frac{2085}{8191}, \frac{2086}{8191} \simeq \frac{4719}{8191}, \frac{4720}{8191}$$
$$\frac{2089}{8191}, \frac{2098}{8191} \simeq \frac{4731}{8191}, \frac{4732}{8191}$$
$$\frac{2090}{8191}, \frac{2097}{8191} \simeq \frac{4739}{8191}, \frac{4740}{8191}$$

$$\frac{2091}{8191}, \frac{2092}{8191} \simeq \frac{4734}{8191}, \frac{4737}{8191}$$
$$\frac{2093}{8191}, \frac{2094}{8191} \simeq \frac{4727}{8191}, \frac{4728}{8191}$$
$$\frac{2095}{8191}, \frac{2096}{8191} \simeq \frac{4741}{8191}, \frac{4730}{8191}$$
$$\frac{2101}{8191}, \frac{2102}{8191} \simeq \frac{3423}{8191}, \frac{3424}{8191}$$
$$\frac{2103}{8191}, \frac{2104}{8191} \simeq \frac{3426}{8191}, \frac{3421}{8191}$$
$$\frac{2106}{8191}, \frac{2113}{8191} \simeq \frac{3476}{8191}, \frac{3475}{8191}$$
$$\frac{2107}{8191}, \frac{2108}{8191} \simeq \frac{3473}{8191}, \frac{3470}{8191}$$
$$\frac{2109}{8191}, \frac{2110}{8191} \simeq \frac{4712}{8191}, \frac{4711}{8191}$$
$$\frac{2111}{8191}, \frac{2112}{8191} \simeq \frac{4714}{8191}, \frac{3477}{8191}$$
$$\frac{2117}{8191}, \frac{2118}{8191} \simeq \frac{3471}{8191}, \frac{3472}{8191}$$
$$\frac{2119}{8191}, \frac{2120}{8191} \simeq \frac{3469}{8191}, \frac{3442}{8191}$$
$$\frac{2123}{8191}, \frac{2124}{8191} \simeq \frac{3294}{8191}, \frac{3297}{8191}$$
$$\frac{2125}{8191}, \frac{2126}{8191} \simeq \frac{3351}{8191}, \frac{3352}{8191}$$
$$\frac{2127}{8191}, \frac{2128}{8191} \simeq \frac{3365}{8191}, \frac{3354}{8191}$$
$$\frac{2131}{8191}, \frac{2148}{8191} \simeq \frac{3321}{8191}, \frac{3334}{8191}$$
$$\frac{2132}{8191}, \frac{2147}{8191} \simeq \frac{3318}{8191}, \frac{3337}{8191}$$
$$\frac{2133}{8191}, \frac{2134}{8191} \simeq \frac{3327}{8191}, \frac{3328}{8191}$$
$$\frac{2135}{8191}, \frac{2136}{8191} \simeq \frac{3325}{8191}, \frac{3330}{8191}$$
$$\frac{2137}{8191}, \frac{2146}{8191} \simeq \frac{3339}{8191}, \frac{3340}{8191}$$
$$\frac{2138}{8191}, \frac{2145}{8191} \simeq \frac{3315}{8191}, \frac{3316}{8191}$$
$$\frac{2139}{8191}, \frac{2140}{8191} \simeq \frac{3313}{8191}, \frac{3310}{8191}$$
$$\frac{2141}{8191}, \frac{2142}{8191} \simeq \frac{3335}{8191}, \frac{3336}{8191}$$
$$\frac{2143}{8191}, \frac{2144}{8191} \simeq \frac{3338}{8191}, \frac{3317}{8191}$$
$$\frac{2149}{8191}, \frac{2150}{8191} \simeq \frac{3375}{8191}, \frac{3376}{8191}$$
$$\frac{2151}{8191}, \frac{2152}{8191} \simeq \frac{4818}{8191}, \frac{4813}{8191}$$
$$\frac{2153}{8191}, \frac{2162}{8191} \simeq \frac{4795}{8191}, \frac{4796}{8191}$$
$$\frac{2154}{8191}, \frac{2161}{8191} \simeq \frac{4803}{8191}, \frac{4804}{8191}$$
$$\frac{2155}{8191}, \frac{2156}{8191} \simeq \frac{4798}{8191}, \frac{4801}{8191}$$
$$\frac{2157}{8191}, \frac{2158}{8191} \simeq \frac{4791}{8191}, \frac{4792}{8191}$$
$$\frac{2159}{8191}, \frac{2160}{8191} \simeq \frac{4805}{8191}, \frac{4794}{8191}$$
$$\frac{2163}{8191}, \frac{2180}{8191} \simeq \frac{4902}{8191}, \frac{4825}{8191}$$
$$\frac{2164}{8191}, \frac{2179}{8191} \simeq \frac{4905}{8191}, \frac{4822}{8191}$$
$$\frac{2165}{8191}, \frac{2166}{8191} \simeq \frac{4895}{8191}, \frac{4896}{8191}$$
$$\frac{2167}{8191}, \frac{2168}{8191} \simeq \frac{4898}{8191}, \frac{4893}{8191}$$
$$\frac{2169}{8191}, \frac{2178}{8191} \simeq \frac{4908}{8191}, \frac{4907}{8191}$$
$$\frac{2170}{8191}, \frac{2177}{8191} \simeq \frac{4820}{8191}, \frac{4819}{8191}$$

$$\frac{2171}{8191}, \frac{2172}{8191} \simeq \frac{4817}{8191}, \frac{4814}{8191}$$
$$\frac{2173}{8191}, \frac{2174}{8191} \simeq \frac{4904}{8191}, \frac{4903}{8191}$$
$$\frac{2175}{8191}, \frac{2176}{8191} \simeq \frac{4906}{8191}, \frac{4821}{8191}$$
$$\frac{2181}{8191}, \frac{2182}{8191} \simeq \frac{4815}{8191}, \frac{4816}{8191}$$
$$\frac{2183}{8191}, \frac{2184}{8191} \simeq \frac{4914}{8191}, \frac{4909}{8191}$$
$$\frac{2187}{8191}, \frac{2188}{8191} \simeq \frac{4894}{8191}, \frac{4897}{8191}$$
$$\frac{2189}{8191}, \frac{2190}{8191} \simeq \frac{4823}{8191}, \frac{4824}{8191}$$
$$\frac{2191}{8191}, \frac{2192}{8191} \simeq \frac{4837}{8191}, \frac{4826}{8191}$$
$$\frac{2195}{8191}, \frac{2196}{8191} \simeq \frac{4537}{8191}, \frac{4550}{8191}$$
$$\frac{2197}{8191}, \frac{2198}{8191} \simeq \frac{4544}{8191}, \frac{4543}{8191}$$
$$\frac{2199}{8191}, \frac{2200}{8191} \simeq \frac{4541}{8191}, \frac{4546}{8191}$$
$$\frac{2201}{8191}, \frac{2210}{8191} \simeq \frac{4555}{8191}, \frac{4660}{8191}$$
$$\frac{2202}{8191}, \frac{2209}{8191} \simeq \frac{4556}{8191}, \frac{4659}{8191}$$
$$\frac{2203}{8191}, \frac{2204}{8191} \simeq \frac{4654}{8191}, \frac{4657}{8191}$$
$$\frac{2205}{8191}, \frac{2206}{8191} \simeq \frac{4551}{8191}, \frac{4552}{8191}$$
$$\frac{2207}{8191}, \frac{2208}{8191} \simeq \frac{4661}{8191}, \frac{4554}{8191}$$
$$\frac{2211}{8191}, \frac{2212}{8191} \simeq \frac{4662}{8191}, \frac{4553}{8191}$$
$$\frac{2213}{8191}, \frac{2214}{8191} \simeq \frac{4591}{8191}, \frac{4592}{8191}$$

$$\frac{2215}{8191}, \frac{2216}{8191} \simeq \frac{4594}{8191}, \frac{4621}{8191}$$
$$\frac{2217}{8191}, \frac{2226}{8191} \simeq \frac{4603}{8191}, \frac{4612}{8191}$$
$$\frac{2218}{8191}, \frac{2225}{8191} \simeq \frac{4604}{8191}, \frac{4611}{8191}$$
$$\frac{2219}{8191}, \frac{2220}{8191} \simeq \frac{4606}{8191}, \frac{4609}{8191}$$
$$\frac{2221}{8191}, \frac{2222}{8191} \simeq \frac{4599}{8191}, \frac{4600}{8191}$$
$$\frac{2223}{8191}, \frac{2224}{8191} \simeq \frac{4602}{8191}, \frac{4613}{8191}$$
$$\frac{2227}{8191}, \frac{2228}{8191} \simeq \frac{4582}{8191}, \frac{4633}{8191}$$
$$\frac{2229}{8191}, \frac{2230}{8191} \simeq \frac{4575}{8191}, \frac{4576}{8191}$$
$$\frac{2231}{8191}, \frac{2232}{8191} \simeq \frac{4573}{8191}, \frac{4578}{8191}$$
$$\frac{2233}{8191}, \frac{2242}{8191} \simeq \frac{4587}{8191}, \frac{4628}{8191}$$
$$\frac{2234}{8191}, \frac{2241}{8191} \simeq \frac{4588}{8191}, \frac{4627}{8191}$$
$$\frac{2235}{8191}, \frac{2236}{8191} \simeq \frac{4622}{8191}, \frac{4625}{8191}$$
$$\frac{2237}{8191}, \frac{2238}{8191} \simeq \frac{4583}{8191}, \frac{4584}{8191}$$
$$\frac{2239}{8191}, \frac{2240}{8191} \simeq \frac{4629}{8191}, \frac{4586}{8191}$$
$$\frac{2243}{8191}, \frac{2244}{8191} \simeq \frac{4585}{8191}, \frac{4630}{8191}$$
$$\frac{2245}{8191}, \frac{2246}{8191} \simeq \frac{4623}{8191}, \frac{4624}{8191}$$
$$\frac{2247}{8191}, \frac{2248}{8191} \simeq \frac{4589}{8191}, \frac{4626}{8191}$$
$$\frac{2249}{8191}, \frac{2322}{8191} \simeq \frac{4699}{8191}, \frac{4700}{8191}$$

$$\frac{2250}{8191},\frac{2257}{8191} \simeq \frac{4707}{8191},\frac{4708}{8191}$$
$$\frac{2251}{8191},\frac{2252}{8191} \simeq \frac{4702}{8191},\frac{4705}{8191}$$
$$\frac{2253}{8191},\frac{2254}{8191} \simeq \frac{3479}{8191},\frac{3480}{8191}$$
$$\frac{2255}{8191},\frac{2256}{8191} \simeq \frac{4709}{8191},\frac{3482}{8191}$$
$$\frac{2258}{8191},\frac{2313}{8191} \simeq \frac{3483}{8191},\frac{3484}{8191}$$
$$\frac{2259}{8191},\frac{2260}{8191} \simeq \frac{3449}{8191},\frac{3462}{8191}$$
$$\frac{2261}{8191},\frac{2262}{8191} \simeq \frac{3455}{8191},\frac{3456}{8191}$$
$$\frac{2263}{8191},\frac{2264}{8191} \simeq \frac{3453}{8191},\frac{3458}{8191}$$
$$\frac{2265}{8191},\frac{2274}{8191} \simeq \frac{3467}{8191},\frac{3468}{8191}$$
$$\frac{2266}{8191},\frac{2273}{8191} \simeq \frac{3443}{8191},\frac{3444}{8191}$$
$$\frac{2267}{8191},\frac{2268}{8191} \simeq \frac{3441}{8191},\frac{3438}{8191}$$
$$\frac{2269}{8191},\frac{2270}{8191} \simeq \frac{3463}{8191},\frac{3464}{8191}$$
$$\frac{2271}{8191},\frac{2272}{8191} \simeq \frac{3466}{8191},\frac{3445}{8191}$$
$$\frac{2275}{8191},\frac{2276}{8191} \simeq \frac{3465}{8191},\frac{3446}{8191}$$
$$\frac{2277}{8191},\frac{2278}{8191} \simeq \frac{3503}{8191},\frac{3504}{8191}$$
$$\frac{2279}{8191},\frac{2280}{8191} \simeq \frac{4685}{8191},\frac{3506}{8191}$$
$$\frac{2281}{8191},\frac{2290}{8191} \simeq \frac{4667}{8191},\frac{4676}{8191}$$
$$\frac{2282}{8191},\frac{2289}{8191} \simeq \frac{4668}{8191},\frac{4675}{8191}$$

$$\frac{2283}{8191},\frac{2284}{8191} \simeq \frac{4670}{8191},\frac{4673}{8191}$$
$$\frac{2285}{8191},\frac{2286}{8191} \simeq \frac{4663}{8191},\frac{4664}{8191}$$
$$\frac{2287}{8191},\frac{2288}{8191} \simeq \frac{4677}{8191},\frac{4666}{8191}$$
$$\frac{2291}{8191},\frac{2292}{8191} \simeq \frac{4697}{8191},\frac{3494}{8191}$$
$$\frac{2293}{8191},\frac{2294}{8191} \simeq \frac{3487}{8191},\frac{3488}{8191}$$
$$\frac{2295}{8191},\frac{2296}{8191} \simeq \frac{3490}{8191},\frac{3485}{8191}$$
$$\frac{2297}{8191},\frac{2306}{8191} \simeq \frac{3500}{8191},\frac{3499}{8191}$$
$$\frac{2298}{8191},\frac{2305}{8191} \simeq \frac{4692}{8191},\frac{4691}{8191}$$
$$\frac{2299}{8191},\frac{2300}{8191} \simeq \frac{4689}{8191},\frac{4686}{8191}$$
$$\frac{2301}{8191},\frac{2302}{8191} \simeq \frac{3496}{8191},\frac{3495}{8191}$$
$$\frac{2303}{8191},\frac{2304}{8191} \simeq \frac{4693}{8191},\frac{3498}{8191}$$
$$\frac{2307}{8191},\frac{2308}{8191} \simeq \frac{4694}{8191},\frac{3497}{8191}$$
$$\frac{2309}{8191},\frac{2310}{8191} \simeq \frac{4687}{8191},\frac{4688}{8191}$$
$$\frac{2311}{8191},\frac{2312}{8191} \simeq \frac{4690}{8191},\frac{3501}{8191}$$
$$\frac{2314}{8191},\frac{2321}{8191} \simeq \frac{3491}{8191},\frac{3492}{8191}$$
$$\frac{2315}{8191},\frac{2316}{8191} \simeq \frac{3489}{8191},\frac{3486}{8191}$$
$$\frac{2317}{8191},\frac{2318}{8191} \simeq \frac{4695}{8191},\frac{4696}{8191}$$
$$\frac{2319}{8191},\frac{2320}{8191} \simeq \frac{4698}{8191},\frac{3493}{8191}$$

$$\frac{2323}{8191}, \frac{2324}{8191} \simeq \frac{4665}{8191}, \frac{4678}{8191}$$
$$\frac{2325}{8191}, \frac{2326}{8191} \simeq \frac{4671}{8191}, \frac{4672}{8191}$$
$$\frac{2327}{8191}, \frac{2328}{8191} \simeq \frac{4669}{8191}, \frac{4674}{8191}$$
$$\frac{2329}{8191}, \frac{2338}{8191} \simeq \frac{4683}{8191}, \frac{4684}{8191}$$
$$\frac{2330}{8191}, \frac{2337}{8191} \simeq \frac{3507}{8191}, \frac{3508}{8191}$$
$$\frac{2331}{8191}, \frac{2332}{8191} \simeq \frac{3505}{8191}, \frac{3502}{8191}$$
$$\frac{2333}{8191}, \frac{2334}{8191} \simeq \frac{4679}{8191}, \frac{4680}{8191}$$
$$\frac{2335}{8191}, \frac{2336}{8191} \simeq \frac{4682}{8191}, \frac{3509}{8191}$$
$$\frac{2339}{8191}, \frac{2340}{8191} \simeq \frac{4681}{8191}, \frac{3510}{8191}$$

B.11 Period 14

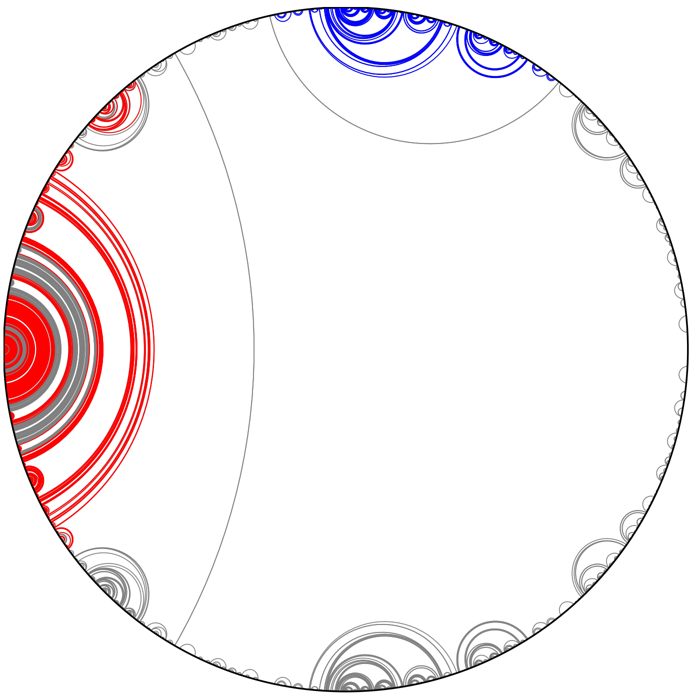

Figure B.11: Leaves μ_q in red and μ_p in blue where $s_{3/7} \perp\!\!\!\perp s_p \simeq s_{1/7} \perp\!\!\!\perp s_q$, for all $\mu_p \in (1/7, 2/7)$ of period 14.

$$\frac{2341}{16383}, \frac{2342}{16383} \simeq \frac{2341}{5461}, \frac{7024}{16383}$$
$$\frac{2343}{16383}, \frac{2344}{16383} \simeq \frac{2351}{5461}, \frac{2342}{5461}$$
$$\frac{2345}{16383}, \frac{2354}{16383} \simeq \frac{2345}{5461}, \frac{2348}{5461}$$
$$\frac{2346}{16383}, \frac{2353}{16383} \simeq \frac{7036}{16383}, \frac{7043}{16383}$$
$$\frac{2347}{16383}, \frac{2348}{16383} \simeq \frac{2346}{5461}, \frac{2347}{5461}$$
$$\frac{2349}{16383}, \frac{2350}{16383} \simeq \frac{7031}{16383}, \frac{2344}{5461}$$
$$\frac{2351}{16383}, \frac{2352}{16383} \simeq \frac{7045}{16383}, \frac{7034}{16383}$$
$$\frac{2355}{16383}, \frac{2356}{16383} \simeq \frac{2355}{5461}, \frac{7270}{16383}$$
$$\frac{2357}{16383}, \frac{2358}{16383} \simeq \frac{2421}{5461}, \frac{7264}{16383}$$
$$\frac{2359}{16383}, \frac{2360}{16383} \simeq \frac{7261}{16383}, \frac{2422}{5461}$$
$$\frac{2361}{16383}, \frac{2370}{16383} \simeq \frac{2425}{5461}, \frac{7060}{16383}$$
$$\frac{2362}{16383}, \frac{2369}{16383} \simeq \frac{2353}{5461}, \frac{7276}{16383}$$
$$\frac{2363}{16383}, \frac{2364}{16383} \simeq \frac{7057}{16383}, \frac{7054}{16383}$$
$$\frac{2365}{16383}, \frac{2366}{16383} \simeq \frac{7271}{16383}, \frac{2424}{5461}$$
$$\frac{2367}{16383}, \frac{2368}{16383} \simeq \frac{7274}{16383}, \frac{7061}{16383}$$
$$\frac{2371}{16383}, \frac{2372}{16383} \simeq \frac{7273}{16383}, \frac{2354}{5461}$$
$$\frac{2373}{16383}, \frac{2374}{16383} \simeq \frac{7055}{16383}, \frac{2352}{5461}$$
$$\frac{2375}{16383}, \frac{2376}{16383} \simeq \frac{7277}{16383}, \frac{7058}{16383}$$
$$\frac{2377}{16383}, \frac{2450}{16383} \simeq \frac{2377}{5461}, \frac{7204}{16383}$$
$$\frac{2378}{16383}, \frac{2385}{16383} \simeq \frac{7132}{16383}, \frac{7139}{16383}$$
$$\frac{2379}{16383}, \frac{2380}{16383} \simeq \frac{2378}{5461}, \frac{2379}{5461}$$
$$\frac{2381}{16383}, \frac{2382}{16383} \simeq \frac{2397}{5461}, \frac{7192}{16383}$$
$$\frac{2383}{16383}, \frac{2384}{16383} \simeq \frac{7141}{16383}, \frac{2398}{5461}$$
$$\frac{2386}{16383}, \frac{2441}{16383} \simeq \frac{2380}{5461}, \frac{7195}{16383}$$
$$\frac{2387}{16383}, \frac{2388}{16383} \simeq \frac{2387}{5461}, \frac{7174}{16383}$$
$$\frac{2389}{16383}, \frac{2390}{16383} \simeq \frac{7168}{16383}, \frac{2389}{5461}$$
$$\frac{2391}{16383}, \frac{2392}{16383} \simeq \frac{7165}{16383}, \frac{2390}{5461}$$
$$\frac{2393}{16383}, \frac{2402}{16383} \simeq \frac{7156}{16383}, \frac{2393}{5461}$$
$$\frac{2394}{16383}, \frac{2401}{16383} \simeq \frac{2385}{5461}, \frac{7180}{16383}$$
$$\frac{2395}{16383}, \frac{2396}{16383} \simeq \frac{7150}{16383}, \frac{7153}{16383}$$
$$\frac{2397}{16383}, \frac{2398}{16383} \simeq \frac{7175}{16383}, \frac{2392}{5461}$$
$$\frac{2399}{16383}, \frac{2400}{16383} \simeq \frac{7157}{16383}, \frac{7178}{16383}$$
$$\frac{2403}{16383}, \frac{2404}{16383} \simeq \frac{2386}{5461}, \frac{7177}{16383}$$
$$\frac{2405}{16383}, \frac{2406}{16383} \simeq \frac{2405}{5461}, \frac{7216}{16383}$$
$$\frac{2407}{16383}, \frac{2408}{16383} \simeq \frac{7117}{16383}, \frac{2406}{5461}$$
$$\frac{2409}{16383}, \frac{2418}{16383} \simeq \frac{7099}{16383}, \frac{7108}{16383}$$

$$\frac{2410}{16383}, \frac{2417}{16383} \simeq \frac{7100}{16383}, \frac{2369}{5461}$$
$$\frac{2411}{16383}, \frac{2412}{16383} \simeq \frac{7102}{16383}, \frac{7105}{16383}$$
$$\frac{2413}{16383}, \frac{2414}{16383} \simeq \frac{7223}{16383}, \frac{7096}{16383}$$
$$\frac{2415}{16383}, \frac{2416}{16383} \simeq \frac{7109}{16383}, \frac{2366}{5461}$$
$$\frac{2419}{16383}, \frac{2420}{16383} \simeq \frac{7129}{16383}, \frac{2402}{5461}$$
$$\frac{2421}{16383}, \frac{2422}{16383} \simeq \frac{7199}{16383}, \frac{2400}{5461}$$
$$\frac{2423}{16383}, \frac{2424}{16383} \simeq \frac{2399}{5461}, \frac{7202}{16383}$$
$$\frac{2425}{16383}, \frac{2434}{16383} \simeq \frac{7211}{16383}, \frac{7124}{16383}$$
$$\frac{2426}{16383}, \frac{2433}{16383} \simeq \frac{7123}{16383}, \frac{2404}{5461}$$
$$\frac{2427}{16383}, \frac{2428}{16383} \simeq \frac{7121}{16383}, \frac{7118}{16383}$$
$$\frac{2429}{16383}, \frac{2430}{16383} \simeq \frac{7207}{16383}, \frac{7208}{16383}$$
$$\frac{2431}{16383}, \frac{2432}{16383} \simeq \frac{7210}{16383}, \frac{2375}{5461}$$
$$\frac{2435}{16383}, \frac{2436}{16383} \simeq \frac{2403}{5461}, \frac{7126}{16383}$$
$$\frac{2437}{16383}, \frac{2438}{16383} \simeq \frac{2373}{5461}, \frac{7120}{16383}$$
$$\frac{2439}{16383}, \frac{2440}{16383} \simeq \frac{7213}{16383}, \frac{2374}{5461}$$
$$\frac{2442}{16383}, \frac{2449}{16383} \simeq \frac{7196}{16383}, \frac{2401}{5461}$$
$$\frac{2443}{16383}, \frac{2444}{16383} \simeq \frac{7198}{16383}, \frac{7201}{16383}$$
$$\frac{2445}{16383}, \frac{2446}{16383} \simeq \frac{7127}{16383}, \frac{2376}{5461}$$
$$\frac{2447}{16383}, \frac{2448}{16383} \simeq \frac{7205}{16383}, \frac{7130}{16383}$$
$$\frac{2452}{16383}, \frac{2579}{16383} \simeq \frac{58}{129}, \frac{71}{129}$$
$$\frac{2453}{16383}, \frac{2454}{16383} \simeq \frac{2453}{5461}, \frac{7360}{16383}$$
$$\frac{2455}{16383}, \frac{2456}{16383} \simeq \frac{7357}{16383}, \frac{2454}{5461}$$
$$\frac{2457}{16383}, \frac{2466}{16383} \simeq \frac{2457}{5461}, \frac{3004}{5461}$$
$$\frac{2458}{16383}, \frac{2465}{16383} \simeq \frac{7372}{16383}, \frac{9011}{16383}$$
$$\frac{2459}{16383}, \frac{2460}{16383} \simeq \frac{3002}{5461}, \frac{3003}{5461}$$
$$\frac{2461}{16383}, \frac{2462}{16383} \simeq \frac{7367}{16383}, \frac{2456}{5461}$$
$$\frac{2463}{16383}, \frac{2464}{16383} \simeq \frac{9013}{16383}, \frac{7370}{16383}$$
$$\frac{2467}{16383}, \frac{2468}{16383} \simeq \frac{9014}{16383}, \frac{7369}{16383}$$
$$\frac{2469}{16383}, \frac{2470}{16383} \simeq \frac{2981}{5461}, \frac{208}{381}$$
$$\frac{2471}{16383}, \frac{2472}{16383} \simeq \frac{2982}{5461}, \frac{2991}{5461}$$
$$\frac{2473}{16383}, \frac{2482}{16383} \simeq \frac{2985}{5461}, \frac{2988}{5461}$$
$$\frac{2474}{16383}, \frac{2481}{16383} \simeq \frac{8956}{16383}, \frac{8963}{16383}$$
$$\frac{2475}{16383}, \frac{2476}{16383} \simeq \frac{2986}{5461}, \frac{2987}{5461}$$
$$\frac{2477}{16383}, \frac{2478}{16383} \simeq \frac{8951}{16383}, \frac{2984}{5461}$$
$$\frac{2479}{16383}, \frac{2480}{16383} \simeq \frac{8954}{16383}, \frac{8965}{16383}$$
$$\frac{2483}{16383}, \frac{2484}{16383} \simeq \frac{2483}{5461}, \frac{2978}{5461}$$

$$\frac{2485}{16383}, \frac{2486}{16383} \simeq \frac{8927}{16383}, \frac{2976}{5461}$$

$$\frac{2487}{16383}, \frac{2488}{16383} \simeq \frac{2975}{5461}, \frac{8930}{16383}$$

$$\frac{2489}{16383}, \frac{2498}{16383} \simeq \frac{8939}{16383}, \frac{8980}{16383}$$

$$\frac{2490}{16383}, \frac{2497}{16383} \simeq \frac{2980}{5461}, \frac{2993}{5461}$$

$$\frac{2491}{16383}, \frac{2492}{16383} \simeq \frac{8974}{16383}, \frac{8945}{16383}$$

$$\frac{2493}{16383}, \frac{2494}{16383} \simeq \frac{8935}{16383}, \frac{8936}{16383}$$

$$\frac{2495}{16383}, \frac{2496}{16383} \simeq \frac{8981}{16383}, \frac{8938}{16383}$$

$$\frac{2499}{16383}, \frac{2500}{16383} \simeq \frac{2979}{5461}, \frac{2994}{5461}$$

$$\frac{2501}{16383}, \frac{2502}{16383} \simeq \frac{8975}{16383}, \frac{2992}{5461}$$

$$\frac{2503}{16383}, \frac{2504}{16383} \simeq \frac{8941}{16383}, \frac{8978}{16383}$$

$$\frac{2505}{16383}, \frac{2578}{16383} \simeq \frac{3017}{5461}, \frac{2444}{5461}$$

$$\frac{2506}{16383}, \frac{2513}{16383} \simeq \frac{7324}{16383}, \frac{9059}{16383}$$

$$\frac{2507}{16383}, \frac{2508}{16383} \simeq \frac{3018}{5461}, \frac{3019}{5461}$$

$$\frac{2509}{16383}, \frac{2510}{16383} \simeq \frac{7319}{16383}, \frac{2440}{5461}$$

$$\frac{2511}{16383}, \frac{2512}{16383} \simeq \frac{9061}{16383}, \frac{7322}{16383}$$

$$\frac{2514}{16383}, \frac{2569}{16383} \simeq \frac{2441}{5461}, \frac{3020}{5461}$$

$$\frac{2515}{16383}, \frac{2516}{16383} \simeq \frac{7289}{16383}, \frac{2434}{5461}$$

$$\frac{2517}{16383}, \frac{2518}{16383} \simeq \frac{7295}{16383}, \frac{2432}{5461}$$

$$\frac{2519}{16383}, \frac{2520}{16383} \simeq \frac{2431}{5461}, \frac{7298}{16383}$$

$$\frac{2521}{16383}, \frac{2530}{16383} \simeq \frac{7307}{16383}, \frac{2428}{5461}$$

$$\frac{2522}{16383}, \frac{2529}{16383} \simeq \frac{7283}{16383}, \frac{2436}{5461}$$

$$\frac{2523}{16383}, \frac{2524}{16383} \simeq \frac{2427}{5461}, \frac{2426}{5461}$$

$$\frac{2525}{16383}, \frac{2526}{16383} \simeq \frac{7303}{16383}, \frac{7304}{16383}$$

$$\frac{2527}{16383}, \frac{2528}{16383} \simeq \frac{7306}{16383}, \frac{7285}{16383}$$

$$\frac{2531}{16383}, \frac{2532}{16383} \simeq \frac{2435}{5461}, \frac{7286}{16383}$$

$$\frac{2533}{16383}, \frac{2534}{16383} \simeq \frac{7343}{16383}, \frac{2448}{5461}$$

$$\frac{2535}{16383}, \frac{2536}{16383} \simeq \frac{9037}{16383}, \frac{7346}{16383}$$

$$\frac{2537}{16383}, \frac{2546}{16383} \simeq \frac{9019}{16383}, \frac{9028}{16383}$$

$$\frac{2538}{16383}, \frac{2545}{16383} \simeq \frac{9020}{16383}, \frac{3009}{5461}$$

$$\frac{2539}{16383}, \frac{2540}{16383} \simeq \frac{9022}{16383}, \frac{9025}{16383}$$

$$\frac{2541}{16383}, \frac{2542}{16383} \simeq \frac{3005}{5461}, \frac{9016}{16383}$$

$$\frac{2543}{16383}, \frac{2544}{16383} \simeq \frac{9029}{16383}, \frac{3006}{5461}$$

$$\frac{2547}{16383}, \frac{2548}{16383} \simeq \frac{9049}{16383}, \frac{7334}{16383}$$

$$\frac{2549}{16383}, \frac{2550}{16383} \simeq \frac{7327}{16383}, \frac{7328}{16383}$$

$$\frac{2551}{16383}, \frac{2552}{16383} \simeq \frac{7330}{16383}, \frac{7325}{16383}$$

$$\frac{2553}{16383}, \frac{2562}{16383} \simeq \frac{9044}{16383}, \frac{7339}{16383}$$

$$\frac{2554}{16383}, \frac{2561}{16383} \simeq \frac{9043}{16383}, \frac{7340}{16383}$$
$$\frac{2555}{16383}, \frac{2556}{16383} \simeq \frac{9041}{16383}, \frac{9038}{16383}$$
$$\frac{2557}{16383}, \frac{2558}{16383} \simeq \frac{7336}{16383}, \frac{2445}{5461}$$
$$\frac{2559}{16383}, \frac{2560}{16383} \simeq \frac{3015}{5461}, \frac{2446}{5461}$$
$$\frac{2563}{16383}, \frac{2564}{16383} \simeq \frac{9046}{16383}, \frac{7337}{16383}$$
$$\frac{2565}{16383}, \frac{2566}{16383} \simeq \frac{3013}{5461}, \frac{9040}{16383}$$
$$\frac{2567}{16383}, \frac{2568}{16383} \simeq \frac{3014}{5461}, \frac{2447}{5461}$$
$$\frac{2570}{16383}, \frac{2577}{16383} \simeq \frac{7331}{16383}, \frac{9052}{16383}$$
$$\frac{2571}{16383}, \frac{2572}{16383} \simeq \frac{2443}{5461}, \frac{2442}{5461}$$
$$\frac{2573}{16383}, \frac{2574}{16383} \simeq \frac{9047}{16383}, \frac{3016}{5461}$$
$$\frac{2575}{16383}, \frac{2576}{16383} \simeq \frac{9050}{16383}, \frac{7333}{16383}$$
$$\frac{2581}{16383}, \frac{2582}{16383} \simeq \frac{9023}{16383}, \frac{3008}{5461}$$
$$\frac{2583}{16383}, \frac{2584}{16383} \simeq \frac{3007}{5461}, \frac{9026}{16383}$$
$$\frac{2585}{16383}, \frac{2594}{16383} \simeq \frac{9035}{16383}, \frac{7348}{16383}$$
$$\frac{2586}{16383}, \frac{2593}{16383} \simeq \frac{2449}{5461}, \frac{3012}{5461}$$
$$\frac{2587}{16383}, \frac{2588}{16383} \simeq \frac{7345}{16383}, \frac{7342}{16383}$$
$$\frac{2589}{16383}, \frac{2590}{16383} \simeq \frac{9031}{16383}, \frac{9032}{16383}$$
$$\frac{2591}{16383}, \frac{2592}{16383} \simeq \frac{9034}{16383}, \frac{7349}{16383}$$

$$\frac{2595}{16383}, \frac{2596}{16383} \simeq \frac{3011}{5461}, \frac{2450}{5461}$$
$$\frac{2597}{16383}, \frac{2598}{16383} \simeq \frac{7279}{16383}, \frac{7280}{16383}$$
$$\frac{2599}{16383}, \frac{2600}{16383} \simeq \frac{7309}{16383}, \frac{7282}{16383}$$
$$\frac{2601}{16383}, \frac{2610}{16383} \simeq \frac{7291}{16383}, \frac{7300}{16383}$$
$$\frac{2602}{16383}, \frac{2609}{16383} \simeq \frac{7292}{16383}, \frac{2433}{5461}$$
$$\frac{2603}{16383}, \frac{2604}{16383} \simeq \frac{7294}{16383}, \frac{7297}{16383}$$
$$\frac{2605}{16383}, \frac{2606}{16383} \simeq \frac{2429}{5461}, \frac{7288}{16383}$$
$$\frac{2607}{16383}, \frac{2608}{16383} \simeq \frac{7301}{16383}, \frac{2430}{5461}$$
$$\frac{2611}{16383}, \frac{2612}{16383} \simeq \frac{7321}{16383}, \frac{9062}{16383}$$
$$\frac{2613}{16383}, \frac{2614}{16383} \simeq \frac{9055}{16383}, \frac{9056}{16383}$$
$$\frac{2615}{16383}, \frac{2616}{16383} \simeq \frac{9058}{16383}, \frac{9053}{16383}$$
$$\frac{2617}{16383}, \frac{2626}{16383} \simeq \frac{9067}{16383}, \frac{7316}{16383}$$
$$\frac{2618}{16383}, \frac{2625}{16383} \simeq \frac{7315}{16383}, \frac{9068}{16383}$$
$$\frac{2619}{16383}, \frac{2620}{16383} \simeq \frac{7313}{16383}, \frac{170}{381}$$
$$\frac{2621}{16383}, \frac{2622}{16383} \simeq \frac{3021}{5461}, \frac{9064}{16383}$$
$$\frac{2623}{16383}, \frac{2624}{16383} \simeq \frac{2439}{5461}, \frac{3022}{5461}$$
$$\frac{2627}{16383}, \frac{2628}{16383} \simeq \frac{9065}{16383}, \frac{7318}{16383}$$
$$\frac{2629}{16383}, \frac{2630}{16383} \simeq \frac{2437}{5461}, \frac{7312}{16383}$$

$$\frac{2631}{16383}, \frac{2632}{16383} \simeq \frac{3023}{5461}, \frac{2438}{5461}$$
$$\frac{2633}{16383}, \frac{3218}{16383} \simeq \frac{2633}{5461}, \frac{2828}{5461}$$
$$\frac{2634}{16383}, \frac{3217}{16383} \simeq \frac{7900}{16383}, \frac{8483}{16383}$$
$$\frac{2635}{16383}, \frac{2636}{16383} \simeq \frac{2634}{5461}, \frac{2827}{5461}$$
$$\frac{2637}{16383}, \frac{2638}{16383} \simeq \frac{184}{381}, \frac{197}{381}$$
$$\frac{2639}{16383}, \frac{2640}{16383} \simeq \frac{8474}{16383}, \frac{7909}{16383}$$
$$\frac{2641}{16383}, \frac{3210}{16383} \simeq \frac{7907}{16383}, \frac{8476}{16383}$$
$$\frac{2642}{16383}, \frac{3209}{16383} \simeq \frac{2636}{5461}, \frac{2825}{5461}$$
$$\frac{2643}{16383}, \frac{2644}{16383} \simeq \frac{8441}{16383}, \frac{2818}{5461}$$
$$\frac{2645}{16383}, \frac{2646}{16383} \simeq \frac{2816}{5461}, \frac{8447}{16383}$$
$$\frac{2647}{16383}, \frac{2648}{16383} \simeq \frac{2815}{5461}, \frac{8450}{16383}$$
$$\frac{2649}{16383}, \frac{2658}{16383} \simeq \frac{7924}{16383}, \frac{8459}{16383}$$
$$\frac{2650}{16383}, \frac{2657}{16383} \simeq \frac{2641}{5461}, \frac{2820}{5461}$$
$$\frac{2651}{16383}, \frac{2652}{16383} \simeq \frac{7918}{16383}, \frac{8465}{16383}$$
$$\frac{2653}{16383}, \frac{2654}{16383} \simeq \frac{8440}{16383}, \frac{8455}{16383}$$
$$\frac{2655}{16383}, \frac{2656}{16383} \simeq \frac{7925}{16383}, \frac{8458}{16383}$$
$$\frac{2659}{16383}, \frac{2660}{16383} \simeq \frac{2642}{5461}, \frac{2819}{5461}$$
$$\frac{2661}{16383}, \frac{2694}{16383} \simeq \frac{7984}{16383}, \frac{8399}{16383}$$

$$\frac{2662}{16383}, \frac{2693}{16383} \simeq \frac{2661}{5461}, \frac{2800}{5461}$$
$$\frac{2663}{16383}, \frac{2664}{16383} \simeq \frac{2662}{5461}, \frac{2799}{5461}$$
$$\frac{2665}{16383}, \frac{2674}{16383} \simeq \frac{2668}{5461}, \frac{2793}{5461}$$
$$\frac{2666}{16383}, \frac{2673}{16383} \simeq \frac{8003}{16383}, \frac{8380}{16383}$$
$$\frac{2667}{16383}, \frac{2668}{16383} \simeq \frac{22}{43}, \frac{21}{43}$$
$$\frac{2669}{16383}, \frac{2670}{16383} \simeq \frac{8008}{16383}, \frac{8375}{16383}$$
$$\frac{2671}{16383}, \frac{2672}{16383} \simeq \frac{8005}{16383}, \frac{8378}{16383}$$
$$\frac{2675}{16383}, \frac{2676}{16383} \simeq \frac{2658}{5461}, \frac{2803}{5461}$$
$$\frac{2677}{16383}, \frac{2678}{16383} \simeq \frac{2805}{5461}, \frac{8416}{16383}$$
$$\frac{2679}{16383}, \frac{2680}{16383} \simeq \frac{2655}{5461}, \frac{2806}{5461}$$
$$\frac{2681}{16383}, \frac{2690}{16383} \simeq \frac{7979}{16383}, \frac{8404}{16383}$$
$$\frac{2682}{16383}, \frac{2689}{16383} \simeq \frac{2660}{5461}, \frac{2801}{5461}$$
$$\frac{2683}{16383}, \frac{2684}{16383} \simeq \frac{7985}{16383}, \frac{8398}{16383}$$
$$\frac{2685}{16383}, \frac{2686}{16383} \simeq \frac{7975}{16383}, \frac{8408}{16383}$$
$$\frac{2687}{16383}, \frac{2688}{16383} \simeq \frac{8405}{16383}, \frac{7978}{16383}$$
$$\frac{2691}{16383}, \frac{2692}{16383} \simeq \frac{2659}{5461}, \frac{2802}{5461}$$
$$\frac{2695}{16383}, \frac{2696}{16383} \simeq \frac{7981}{16383}, \frac{8402}{16383}$$
$$\frac{2697}{16383}, \frac{3154}{16383} \simeq \frac{7963}{16383}, \frac{8420}{16383}$$

$$\frac{2698}{16383},\frac{3153}{16383} \simeq \frac{7964}{16383},\frac{8419}{16383}$$
$$\frac{2699}{16383},\frac{2700}{16383} \simeq \frac{7966}{16383},\frac{8417}{16383}$$
$$\frac{2701}{16383},\frac{2702}{16383} \simeq \frac{7976}{16383},\frac{8407}{16383}$$
$$\frac{2703}{16383},\frac{2704}{16383} \simeq \frac{8410}{16383},\frac{7973}{16383}$$
$$\frac{2705}{16383},\frac{3146}{16383} \simeq \frac{2657}{5461},\frac{2804}{5461}$$
$$\frac{2706}{16383},\frac{3145}{16383} \simeq \frac{7972}{16383},\frac{8411}{16383}$$
$$\frac{2707}{16383},\frac{2708}{16383} \simeq \frac{2707}{5461},\frac{2754}{5461}$$
$$\frac{2710}{16383},\frac{2837}{16383} \simeq \frac{64}{129},\frac{65}{129}$$
$$\frac{2711}{16383},\frac{2712}{16383} \simeq \frac{8125}{16383},\frac{8258}{16383}$$
$$\frac{2713}{16383},\frac{2722}{16383} \simeq \frac{2713}{5461},\frac{2748}{5461}$$
$$\frac{2714}{16383},\frac{2721}{16383} \simeq \frac{8140}{16383},\frac{8243}{16383}$$
$$\frac{2715}{16383},\frac{2716}{16383} \simeq \frac{2715}{5461},\frac{2746}{5461}$$
$$\frac{2717}{16383},\frac{2718}{16383} \simeq \frac{8135}{16383},\frac{8248}{16383}$$
$$\frac{2719}{16383},\frac{2720}{16383} \simeq \frac{8138}{16383},\frac{8245}{16383}$$
$$\frac{2723}{16383},\frac{2724}{16383} \simeq \frac{8137}{16383},\frac{8246}{16383}$$
$$\frac{2725}{16383},\frac{2758}{16383} \simeq \frac{8207}{16383},\frac{8176}{16383}$$
$$\frac{2726}{16383},\frac{2757}{16383} \simeq \frac{2736}{5461},\frac{2725}{5461}$$
$$\frac{2727}{16383},\frac{2728}{16383} \simeq \frac{2726}{5461},\frac{2735}{5461}$$
$$\frac{2729}{16383},\frac{2738}{16383} \simeq \frac{2732}{5461},\frac{2729}{5461}$$
$$\frac{2730}{16383},\frac{2737}{16383} \simeq \frac{8195}{16383},\frac{8188}{16383}$$
$$\frac{2731}{16383},\frac{2732}{16383} \simeq \frac{2731}{5461},\frac{2730}{5461}$$
$$\frac{2733}{16383},\frac{2734}{16383} \simeq \frac{8200}{16383},\frac{8183}{16383}$$
$$\frac{2735}{16383},\frac{2736}{16383} \simeq \frac{8186}{16383},\frac{8197}{16383}$$
$$\frac{2739}{16383},\frac{2740}{16383} \simeq \frac{2722}{5461},\frac{2739}{5461}$$
$$\frac{2741}{16383},\frac{2742}{16383} \simeq \frac{2720}{5461},\frac{2741}{5461}$$
$$\frac{2743}{16383},\frac{2744}{16383} \simeq \frac{2719}{5461},\frac{2742}{5461}$$
$$\frac{2745}{16383},\frac{2754}{16383} \simeq \frac{8171}{16383},\frac{8212}{16383}$$
$$\frac{2746}{16383},\frac{2753}{16383} \simeq \frac{2724}{5461},\frac{2737}{5461}$$
$$\frac{2747}{16383},\frac{2748}{16383} \simeq \frac{8177}{16383},\frac{8206}{16383}$$
$$\frac{2749}{16383},\frac{2750}{16383} \simeq \frac{8167}{16383},\frac{8216}{16383}$$
$$\frac{2751}{16383},\frac{2752}{16383} \simeq \frac{190}{381},\frac{191}{381}$$
$$\frac{2755}{16383},\frac{2756}{16383} \simeq \frac{2723}{5461},\frac{2738}{5461}$$
$$\frac{2759}{16383},\frac{2760}{16383} \simeq \frac{8173}{16383},\frac{8210}{16383}$$
$$\frac{2761}{16383},\frac{2834}{16383} \simeq \frac{2700}{5461},\frac{2761}{5461}$$
$$\frac{2762}{16383},\frac{2833}{16383} \simeq \frac{8099}{16383},\frac{8284}{16383}$$
$$\frac{2763}{16383},\frac{2764}{16383} \simeq \frac{2699}{5461},\frac{2762}{5461}$$

$$\frac{2765}{16383}, \frac{2766}{16383} \simeq \frac{8087}{16383}, \frac{8296}{16383}$$
$$\frac{2767}{16383}, \frac{2768}{16383} \simeq \frac{8090}{16383}, \frac{8293}{16383}$$
$$\frac{2769}{16383}, \frac{2826}{16383} \simeq \frac{8092}{16383}, \frac{8291}{16383}$$
$$\frac{2770}{16383}, \frac{2825}{16383} \simeq \frac{2697}{5461}, \frac{2764}{5461}$$
$$\frac{2771}{16383}, \frac{2772}{16383} \simeq \frac{8057}{16383}, \frac{8326}{16383}$$
$$\frac{2773}{16383}, \frac{2774}{16383} \simeq \frac{8063}{16383}, \frac{2688}{5461}$$
$$\frac{2775}{16383}, \frac{2776}{16383} \simeq \frac{8066}{16383}, \frac{8317}{16383}$$
$$\frac{2777}{16383}, \frac{2786}{16383} \simeq \frac{2684}{5461}, \frac{2777}{5461}$$
$$\frac{2778}{16383}, \frac{2785}{16383} \simeq \frac{8051}{16383}, \frac{8332}{16383}$$
$$\frac{2779}{16383}, \frac{2780}{16383} \simeq \frac{2682}{5461}, \frac{2779}{5461}$$
$$\frac{2781}{16383}, \frac{2782}{16383} \simeq \frac{8056}{16383}, \frac{8327}{16383}$$
$$\frac{2783}{16383}, \frac{2784}{16383} \simeq \frac{8053}{16383}, \frac{8330}{16383}$$
$$\frac{2787}{16383}, \frac{2788}{16383} \simeq \frac{8054}{16383}, \frac{8329}{16383}$$
$$\frac{2789}{16383}, \frac{2822}{16383} \simeq \frac{2704}{5461}, \frac{2757}{5461}$$
$$\frac{2790}{16383}, \frac{2821}{16383} \simeq \frac{8111}{16383}, \frac{8272}{16383}$$
$$\frac{2791}{16383}, \frac{2792}{16383} \simeq \frac{8114}{16383}, \frac{8269}{16383}$$
$$\frac{2793}{16383}, \frac{2802}{16383} \simeq \frac{8132}{16383}, \frac{8251}{16383}$$
$$\frac{2794}{16383}, \frac{2801}{16383} \simeq \frac{8131}{16383}, \frac{8252}{16383}$$
$$\frac{2795}{16383}, \frac{2796}{16383} \simeq \frac{8126}{16383}, \frac{8257}{16383}$$
$$\frac{2797}{16383}, \frac{2798}{16383} \simeq \frac{2712}{5461}, \frac{2749}{5461}$$
$$\frac{2799}{16383}, \frac{2800}{16383} \simeq \frac{2711}{5461}, \frac{2750}{5461}$$
$$\frac{2803}{16383}, \frac{2804}{16383} \simeq \frac{8102}{16383}, \frac{8281}{16383}$$
$$\frac{2805}{16383}, \frac{2806}{16383} \simeq \frac{8096}{16383}, \frac{8287}{16383}$$
$$\frac{2807}{16383}, \frac{2808}{16383} \simeq \frac{8093}{16383}, \frac{8290}{16383}$$
$$\frac{2809}{16383}, \frac{2818}{16383} \simeq \frac{8107}{16383}, \frac{8276}{16383}$$
$$\frac{2810}{16383}, \frac{2817}{16383} \simeq \frac{8108}{16383}, \frac{8275}{16383}$$
$$\frac{2811}{16383}, \frac{2812}{16383} \simeq \frac{8113}{16383}, \frac{8270}{16383}$$
$$\frac{2813}{16383}, \frac{2814}{16383} \simeq \frac{2701}{5461}, \frac{2760}{5461}$$
$$\frac{2815}{16383}, \frac{2816}{16383} \simeq \frac{2759}{5461}, \frac{2702}{5461}$$
$$\frac{2819}{16383}, \frac{2820}{16383} \simeq \frac{8278}{16383}, \frac{8105}{16383}$$
$$\frac{2823}{16383}, \frac{2824}{16383} \simeq \frac{2703}{5461}, \frac{2758}{5461}$$
$$\frac{2827}{16383}, \frac{2828}{16383} \simeq \frac{2698}{5461}, \frac{2763}{5461}$$
$$\frac{2829}{16383}, \frac{2830}{16383} \simeq \frac{8104}{16383}, \frac{8279}{16383}$$
$$\frac{2831}{16383}, \frac{2832}{16383} \simeq \frac{8101}{16383}, \frac{8282}{16383}$$
$$\frac{2835}{16383}, \frac{2836}{16383} \simeq \frac{8134}{16383}, \frac{8249}{16383}$$
$$\frac{2839}{16383}, \frac{2840}{16383} \simeq \frac{2710}{5461}, \frac{2751}{5461}$$

$$\frac{2841}{16383}, \frac{2850}{16383} \simeq \frac{8116}{16383}, \frac{8267}{16383}$$
$$\frac{2842}{16383}, \frac{2849}{16383} \simeq \frac{2705}{5461}, \frac{2756}{5461}$$
$$\frac{2843}{16383}, \frac{2844}{16383} \simeq \frac{8110}{16383}, \frac{8273}{16383}$$
$$\frac{2845}{16383}, \frac{2846}{16383} \simeq \frac{8120}{16383}, \frac{8263}{16383}$$
$$\frac{2847}{16383}, \frac{2848}{16383} \simeq \frac{8266}{16383}, \frac{8117}{16383}$$
$$\frac{2851}{16383}, \frac{2852}{16383} \simeq \frac{2706}{5461}, \frac{2755}{5461}$$
$$\frac{2853}{16383}, \frac{3142}{16383} \simeq \frac{7823}{16383}, \frac{2608}{5461}$$
$$\frac{2854}{16383}, \frac{3141}{16383} \simeq \frac{2853}{5461}, \frac{8560}{16383}$$
$$\frac{2855}{16383}, \frac{2856}{16383} \simeq \frac{2854}{5461}, \frac{2607}{5461}$$
$$\frac{2857}{16383}, \frac{3122}{16383} \simeq \frac{2601}{5461}, \frac{2604}{5461}$$
$$\frac{2858}{16383}, \frac{3121}{16383} \simeq \frac{7804}{16383}, \frac{7811}{16383}$$
$$\frac{2859}{16383}, \frac{2860}{16383} \simeq \frac{2602}{5461}, \frac{2603}{5461}$$
$$\frac{2861}{16383}, \frac{2862}{16383} \simeq \frac{7816}{16383}, \frac{8567}{16383}$$
$$\frac{2863}{16383}, \frac{2864}{16383} \simeq \frac{8570}{16383}, \frac{8581}{16383}$$
$$\frac{2865}{16383}, \frac{3114}{16383} \simeq \frac{8572}{16383}, \frac{8579}{16383}$$
$$\frac{2866}{16383}, \frac{3113}{16383} \simeq \frac{2857}{5461}, \frac{2860}{5461}$$
$$\frac{2867}{16383}, \frac{2868}{16383} \simeq \frac{2594}{5461}, \frac{2867}{5461}$$
$$\frac{2869}{16383}, \frac{2870}{16383} \simeq \frac{2869}{5461}, \frac{8608}{16383}$$
$$\frac{2871}{16383}, \frac{2872}{16383} \simeq \frac{2591}{5461}, \frac{2870}{5461}$$
$$\frac{2873}{16383}, \frac{2882}{16383} \simeq \frac{7787}{16383}, \frac{8596}{16383}$$
$$\frac{2874}{16383}, \frac{2881}{16383} \simeq \frac{2596}{5461}, \frac{2865}{5461}$$
$$\frac{2875}{16383}, \frac{2876}{16383} \simeq \frac{7793}{16383}, \frac{8590}{16383}$$
$$\frac{2877}{16383}, \frac{2878}{16383} \simeq \frac{181}{381}, \frac{200}{381}$$
$$\frac{2879}{16383}, \frac{2880}{16383} \simeq \frac{8597}{16383}, \frac{7786}{16383}$$
$$\frac{2883}{16383}, \frac{2884}{16383} \simeq \frac{2866}{5461}, \frac{2595}{5461}$$
$$\frac{2885}{16383}, \frac{3110}{16383} \simeq \frac{2597}{5461}, \frac{7792}{16383}$$
$$\frac{2886}{16383}, \frac{3109}{16383} \simeq \frac{8591}{16383}, \frac{2864}{5461}$$
$$\frac{2887}{16383}, \frac{2888}{16383} \simeq \frac{8594}{16383}, \frac{7789}{16383}$$
$$\frac{2889}{16383}, \frac{2962}{16383} \simeq \frac{7643}{16383}, \frac{8740}{16383}$$
$$\frac{2890}{16383}, \frac{2961}{16383} \simeq \frac{2548}{5461}, \frac{2913}{5461}$$
$$\frac{2891}{16383}, \frac{2956}{16383} \simeq \frac{8734}{16383}, \frac{8737}{16383}$$
$$\frac{2892}{16383}, \frac{2955}{16383} \simeq \frac{2890}{5461}, \frac{2891}{5461}$$
$$\frac{2893}{16383}, \frac{2894}{16383} \simeq \frac{2893}{5461}, \frac{8728}{16383}$$
$$\frac{2895}{16383}, \frac{2896}{16383} \simeq \frac{8677}{16383}, \frac{2910}{5461}$$
$$\frac{2897}{16383}, \frac{2954}{16383} \simeq \frac{8675}{16383}, \frac{8732}{16383}$$
$$\frac{2898}{16383}, \frac{2953}{16383} \simeq \frac{2892}{5461}, \frac{8731}{16383}$$

$\frac{2899}{16383}, \frac{2900}{16383}$	$\simeq$	$\frac{2899}{5461}, \frac{8710}{16383}$	$\frac{2933}{16383}, \frac{2934}{16383}$	$\simeq$	$\frac{8671}{16383}, \frac{8672}{16383}$
$\frac{2901}{16383}, \frac{2902}{16383}$	$\simeq$	$\frac{8704}{16383}, \frac{2901}{5461}$	$\frac{2935}{16383}, \frac{2936}{16383}$	$\simeq$	$\frac{2911}{5461}, \frac{8674}{16383}$
$\frac{2903}{16383}, \frac{2904}{16383}$	$\simeq$	$\frac{8701}{16383}, \frac{2902}{5461}$	$\frac{2937}{16383}, \frac{2946}{16383}$	$\simeq$	$\frac{8747}{16383}, \frac{7636}{16383}$
$\frac{2905}{16383}, \frac{2914}{16383}$	$\simeq$	$\frac{8692}{16383}, \frac{2905}{5461}$	$\frac{2938}{16383}, \frac{2945}{16383}$	$\simeq$	$\frac{2545}{5461}, \frac{2916}{5461}$
$\frac{2906}{16383}, \frac{2913}{16383}$	$\simeq$	$\frac{2897}{5461}, \frac{8716}{16383}$	$\frac{2939}{16383}, \frac{2940}{16383}$	$\simeq$	$\frac{8753}{16383}, \frac{7630}{16383}$
$\frac{2907}{16383}, \frac{2908}{16383}$	$\simeq$	$\frac{202}{381}, \frac{2907}{5461}$	$\frac{2941}{16383}, \frac{2942}{16383}$	$\simeq$	$\frac{7640}{16383}, \frac{7639}{16383}$
$\frac{2909}{16383}, \frac{2910}{16383}$	$\simeq$	$\frac{8695}{16383}, \frac{2904}{5461}$	$\frac{2943}{16383}, \frac{2944}{16383}$	$\simeq$	$\frac{8746}{16383}, \frac{7637}{16383}$
$\frac{2911}{16383}, \frac{2912}{16383}$	$\simeq$	$\frac{8693}{16383}, \frac{8714}{16383}$	$\frac{2947}{16383}, \frac{2948}{16383}$	$\simeq$	$\frac{2915}{5461}, \frac{2546}{5461}$
$\frac{2915}{16383}, \frac{2916}{16383}$	$\simeq$	$\frac{2898}{5461}, \frac{8713}{16383}$	$\frac{2951}{16383}, \frac{2952}{16383}$	$\simeq$	$\frac{8749}{16383}, \frac{7634}{16383}$
$\frac{2917}{16383}, \frac{2950}{16383}$	$\simeq$	$\frac{7631}{16383}, \frac{2544}{5461}$	$\frac{2957}{16383}, \frac{2958}{16383}$	$\simeq$	$\frac{8743}{16383}, \frac{8744}{16383}$
$\frac{2918}{16383}, \frac{2949}{16383}$	$\simeq$	$\frac{2917}{5461}, \frac{8752}{16383}$	$\frac{2959}{16383}, \frac{2960}{16383}$	$\simeq$	$\frac{8741}{16383}, \frac{7642}{16383}$
$\frac{2919}{16383}, \frac{2920}{16383}$	$\simeq$	$\frac{2543}{5461}, \frac{2918}{5461}$	$\frac{2963}{16383}, \frac{2964}{16383}$	$\simeq$	$\frac{7865}{16383}, \frac{8518}{16383}$
$\frac{2921}{16383}, \frac{2930}{16383}$	$\simeq$	$\frac{23}{43}, \frac{20}{43}$	$\frac{2965}{16383}, \frac{2966}{16383}$	$\simeq$	$\frac{2837}{5461}, \frac{8512}{16383}$
$\frac{2922}{16383}, \frac{2929}{16383}$	$\simeq$	$\frac{8764}{16383}, \frac{8771}{16383}$	$\frac{2968}{16383}, \frac{3095}{16383}$	$\simeq$	$\frac{62}{129}, \frac{67}{129}$
$\frac{2923}{16383}, \frac{2924}{16383}$	$\simeq$	$\frac{2922}{5461}, \frac{2923}{5461}$	$\frac{2969}{16383}, \frac{2978}{16383}$	$\simeq$	$\frac{7883}{16383}, \frac{8500}{16383}$
$\frac{2925}{16383}, \frac{2926}{16383}$	$\simeq$	$\frac{2925}{5461}, \frac{8776}{16383}$	$\frac{2970}{16383}, \frac{2977}{16383}$	$\simeq$	$\frac{2628}{5461}, \frac{2833}{5461}$
$\frac{2927}{16383}, \frac{2928}{16383}$	$\simeq$	$\frac{8773}{16383}, \frac{7610}{16383}$	$\frac{2971}{16383}, \frac{2972}{16383}$	$\simeq$	$\frac{7889}{16383}, \frac{8494}{16383}$
$\frac{2931}{16383}, \frac{2932}{16383}$	$\simeq$	$\frac{2547}{5461}, \frac{2914}{5461}$	$\frac{2973}{16383}, \frac{2974}{16383}$	$\simeq$	$\frac{7879}{16383}, \frac{8504}{16383}$

$$\frac{2975}{16383}, \frac{2976}{16383} \simeq \frac{8501}{16383}, \frac{7882}{16383}$$
$$\frac{2979}{16383}, \frac{2980}{16383} \simeq \frac{2834}{5461}, \frac{2627}{5461}$$
$$\frac{2981}{16383}, \frac{3014}{16383} \simeq \frac{7951}{16383}, \frac{8432}{16383}$$
$$\frac{2982}{16383}, \frac{3013}{16383} \simeq \frac{7952}{16383}, \frac{8431}{16383}$$
$$\frac{2983}{16383}, \frac{2984}{16383} \simeq \frac{7949}{16383}, \frac{8434}{16383}$$
$$\frac{2985}{16383}, \frac{2994}{16383} \simeq \frac{7931}{16383}, \frac{7940}{16383}$$
$$\frac{2986}{16383}, \frac{2993}{16383} \simeq \frac{2644}{5461}, \frac{7939}{16383}$$
$$\frac{2987}{16383}, \frac{2988}{16383} \simeq \frac{7934}{16383}, \frac{7937}{16383}$$
$$\frac{2989}{16383}, \frac{2990}{16383} \simeq \frac{7927}{16383}, \frac{8456}{16383}$$
$$\frac{2991}{16383}, \frac{2992}{16383} \simeq \frac{2647}{5461}, \frac{7930}{16383}$$
$$\frac{2995}{16383}, \frac{2996}{16383} \simeq \frac{7961}{16383}, \frac{8422}{16383}$$
$$\frac{2997}{16383}, \frac{2998}{16383} \simeq \frac{7967}{16383}, \frac{2656}{5461}$$
$$\frac{2999}{16383}, \frac{3000}{16383} \simeq \frac{8413}{16383}, \frac{7970}{16383}$$
$$\frac{3001}{16383}, \frac{3010}{16383} \simeq \frac{2809}{5461}, \frac{2652}{5461}$$
$$\frac{3002}{16383}, \frac{3009}{16383} \simeq \frac{185}{381}, \frac{196}{381}$$
$$\frac{3003}{16383}, \frac{3004}{16383} \simeq \frac{2811}{5461}, \frac{2650}{5461}$$
$$\frac{3005}{16383}, \frac{3006}{16383} \simeq \frac{8423}{16383}, \frac{7960}{16383}$$
$$\frac{3007}{16383}, \frac{3008}{16383} \simeq \frac{8426}{16383}, \frac{7957}{16383}$$

$$\frac{3011}{16383}, \frac{3012}{16383} \simeq \frac{8425}{16383}, \frac{7958}{16383}$$
$$\frac{3015}{16383}, \frac{3016}{16383} \simeq \frac{8429}{16383}, \frac{7954}{16383}$$
$$\frac{3017}{16383}, \frac{3090}{16383} \simeq \frac{8539}{16383}, \frac{7844}{16383}$$
$$\frac{3018}{16383}, \frac{3089}{16383} \simeq \frac{7843}{16383}, \frac{8540}{16383}$$
$$\frac{3019}{16383}, \frac{3084}{16383} \simeq \frac{7841}{16383}, \frac{8542}{16383}$$
$$\frac{3020}{16383}, \frac{3083}{16383} \simeq \frac{7838}{16383}, \frac{8545}{16383}$$
$$\frac{3021}{16383}, \frac{3022}{16383} \simeq \frac{7831}{16383}, \frac{8552}{16383}$$
$$\frac{3023}{16383}, \frac{3024}{16383} \simeq \frac{8549}{16383}, \frac{7834}{16383}$$
$$\frac{3025}{16383}, \frac{3082}{16383} \simeq \frac{2849}{5461}, \frac{2612}{5461}$$
$$\frac{3026}{16383}, \frac{3081}{16383} \simeq \frac{7835}{16383}, \frac{8548}{16383}$$
$$\frac{3027}{16383}, \frac{3028}{16383} \simeq \frac{7801}{16383}, \frac{8582}{16383}$$
$$\frac{3029}{16383}, \frac{3030}{16383} \simeq \frac{8575}{16383}, \frac{8576}{16383}$$
$$\frac{3031}{16383}, \frac{3032}{16383} \simeq \frac{8573}{16383}, \frac{8578}{16383}$$
$$\frac{3033}{16383}, \frac{3042}{16383} \simeq \frac{8587}{16383}, \frac{7796}{16383}$$
$$\frac{3034}{16383}, \frac{3041}{16383} \simeq \frac{7795}{16383}, \frac{8588}{16383}$$
$$\frac{3035}{16383}, \frac{3036}{16383} \simeq \frac{8593}{16383}, \frac{7790}{16383}$$
$$\frac{3037}{16383}, \frac{3038}{16383} \simeq \frac{2861}{5461}, \frac{2600}{5461}$$
$$\frac{3039}{16383}, \frac{3040}{16383} \simeq \frac{2862}{5461}, \frac{2599}{5461}$$

$$\frac{3043}{16383}, \frac{3044}{16383} \simeq \frac{8585}{16383}, \frac{7798}{16383}$$
$$\frac{3045}{16383}, \frac{3078}{16383} \simeq \frac{8527}{16383}, \frac{8528}{16383}$$
$$\frac{3046}{16383}, \frac{3077}{16383} \simeq \frac{7855}{16383}, \frac{7856}{16383}$$
$$\frac{3047}{16383}, \frac{3048}{16383} \simeq \frac{8525}{16383}, \frac{7858}{16383}$$
$$\frac{3049}{16383}, \frac{3058}{16383} \simeq \frac{8507}{16383}, \frac{7876}{16383}$$
$$\frac{3050}{16383}, \frac{3057}{16383} \simeq \frac{2625}{5461}, \frac{2836}{5461}$$
$$\frac{3051}{16383}, \frac{3052}{16383} \simeq \frac{7873}{16383}, \frac{7870}{16383}$$
$$\frac{3053}{16383}, \frac{3054}{16383} \simeq \frac{8503}{16383}, \frac{7880}{16383}$$
$$\frac{3055}{16383}, \frac{3056}{16383} \simeq \frac{8506}{16383}, \frac{7877}{16383}$$
$$\frac{3059}{16383}, \frac{3060}{16383} \simeq \frac{8537}{16383}, \frac{7846}{16383}$$
$$\frac{3061}{16383}, \frac{3062}{16383} \simeq \frac{8543}{16383}, \frac{2848}{5461}$$
$$\frac{3063}{16383}, \frac{3064}{16383} \simeq \frac{8546}{16383}, \frac{7837}{16383}$$
$$\frac{3065}{16383}, \frac{3074}{16383} \simeq \frac{2844}{5461}, \frac{2617}{5461}$$
$$\frac{3066}{16383}, \frac{3073}{16383} \simeq \frac{8531}{16383}, \frac{7852}{16383}$$
$$\frac{3067}{16383}, \frac{3068}{16383} \simeq \frac{2842}{5461}, \frac{2619}{5461}$$
$$\frac{3069}{16383}, \frac{3070}{16383} \simeq \frac{8536}{16383}, \frac{7847}{16383}$$
$$\frac{3071}{16383}, \frac{3072}{16383} \simeq \frac{8533}{16383}, \frac{7850}{16383}$$
$$\frac{3075}{16383}, \frac{3076}{16383} \simeq \frac{8534}{16383}, \frac{7849}{16383}$$
$$\frac{3079}{16383}, \frac{3080}{16383} \simeq \frac{8530}{16383}, \frac{7853}{16383}$$
$$\frac{3085}{16383}, \frac{3086}{16383} \simeq \frac{2845}{5461}, \frac{2616}{5461}$$
$$\frac{3087}{16383}, \frac{3088}{16383} \simeq \frac{2846}{5461}, \frac{2615}{5461}$$
$$\frac{3091}{16383}, \frac{3092}{16383} \simeq \frac{2835}{5461}, \frac{2626}{5461}$$
$$\frac{3093}{16383}, \frac{3094}{16383} \simeq \frac{7871}{16383}, \frac{2624}{5461}$$
$$\frac{3097}{16383}, \frac{3106}{16383} \simeq \frac{2841}{5461}, \frac{2620}{5461}$$
$$\frac{3098}{16383}, \frac{3105}{16383} \simeq \frac{7859}{16383}, \frac{8524}{16383}$$
$$\frac{3099}{16383}, \frac{3100}{16383} \simeq \frac{2843}{5461}, \frac{2618}{5461}$$
$$\frac{3101}{16383}, \frac{3102}{16383} \simeq \frac{8519}{16383}, \frac{7864}{16383}$$
$$\frac{3103}{16383}, \frac{3104}{16383} \simeq \frac{8522}{16383}, \frac{7861}{16383}$$
$$\frac{3107}{16383}, \frac{3108}{16383} \simeq \frac{8521}{16383}, \frac{7862}{16383}$$
$$\frac{3111}{16383}, \frac{3112}{16383} \simeq \frac{2863}{5461}, \frac{2598}{5461}$$
$$\frac{3115}{16383}, \frac{3116}{16383} \simeq \frac{2858}{5461}, \frac{2859}{5461}$$
$$\frac{3117}{16383}, \frac{3118}{16383} \simeq \frac{7799}{16383}, \frac{8584}{16383}$$
$$\frac{3119}{16383}, \frac{3120}{16383} \simeq \frac{7813}{16383}, \frac{7802}{16383}$$
$$\frac{3123}{16383}, \frac{3124}{16383} \simeq \frac{2850}{5461}, \frac{2611}{5461}$$
$$\frac{3125}{16383}, \frac{3126}{16383} \simeq \frac{2613}{5461}, \frac{7840}{16383}$$
$$\frac{3127}{16383}, \frac{3128}{16383} \simeq \frac{2847}{5461}, \frac{2614}{5461}$$

$$\frac{3129}{16383}, \frac{3138}{16383} \simeq \frac{8555}{16383}, \frac{7828}{16383}$$
$$\frac{3130}{16383}, \frac{3137}{16383} \simeq \frac{2609}{5461}, \frac{2852}{5461}$$
$$\frac{3131}{16383}, \frac{3132}{16383} \simeq \frac{8561}{16383}, \frac{7822}{16383}$$
$$\frac{3133}{16383}, \frac{3134}{16383} \simeq \frac{8551}{16383}, \frac{7832}{16383}$$
$$\frac{3135}{16383}, \frac{3136}{16383} \simeq \frac{7829}{16383}, \frac{8554}{16383}$$
$$\frac{3139}{16383}, \frac{3140}{16383} \simeq \frac{2851}{5461}, \frac{2610}{5461}$$
$$\frac{3143}{16383}, \frac{3144}{16383} \simeq \frac{199}{381}, \frac{182}{381}$$
$$\frac{3147}{16383}, \frac{3148}{16383} \simeq \frac{7969}{16383}, \frac{8414}{16383}$$
$$\frac{3149}{16383}, \frac{3150}{16383} \simeq \frac{2653}{5461}, \frac{2808}{5461}$$
$$\frac{3151}{16383}, \frac{3152}{16383} \simeq \frac{2807}{5461}, \frac{2654}{5461}$$
$$\frac{3155}{16383}, \frac{3156}{16383} \simeq \frac{2643}{5461}, \frac{7942}{16383}$$
$$\frac{3157}{16383}, \frac{3158}{16383} \simeq \frac{2645}{5461}, \frac{7936}{16383}$$
$$\frac{3159}{16383}, \frac{3160}{16383} \simeq \frac{7933}{16383}, \frac{2646}{5461}$$
$$\frac{3161}{16383}, \frac{3170}{16383} \simeq \frac{2649}{5461}, \frac{2812}{5461}$$
$$\frac{3162}{16383}, \frac{3169}{16383} \simeq \frac{7948}{16383}, \frac{8435}{16383}$$
$$\frac{3163}{16383}, \frac{3164}{16383} \simeq \frac{2651}{5461}, \frac{2810}{5461}$$
$$\frac{3165}{16383}, \frac{3166}{16383} \simeq \frac{7928}{16383}, \frac{7943}{16383}$$
$$\frac{3167}{16383}, \frac{3168}{16383} \simeq \frac{8437}{16383}, \frac{7946}{16383}$$
$$\frac{3171}{16383}, \frac{3172}{16383} \simeq \frac{7945}{16383}, \frac{8438}{16383}$$
$$\frac{3173}{16383}, \frac{3174}{16383} \simeq \frac{2629}{5461}, \frac{7888}{16383}$$
$$\frac{3175}{16383}, \frac{3176}{16383} \simeq \frac{8498}{16383}, \frac{7885}{16383}$$
$$\frac{3177}{16383}, \frac{3186}{16383} \simeq \frac{7867}{16383}, \frac{8516}{16383}$$
$$\frac{3178}{16383}, \frac{3185}{16383} \simeq \frac{7868}{16383}, \frac{8515}{16383}$$
$$\frac{3179}{16383}, \frac{3180}{16383} \simeq \frac{8510}{16383}, \frac{8513}{16383}$$
$$\frac{3181}{16383}, \frac{3182}{16383} \simeq \frac{2621}{5461}, \frac{2840}{5461}$$
$$\frac{3183}{16383}, \frac{3184}{16383} \simeq \frac{2839}{5461}, \frac{2622}{5461}$$
$$\frac{3187}{16383}, \frac{3188}{16383} \simeq \frac{8486}{16383}, \frac{7897}{16383}$$
$$\frac{3189}{16383}, \frac{3190}{16383} \simeq \frac{7903}{16383}, \frac{7904}{16383}$$
$$\frac{3191}{16383}, \frac{3192}{16383} \simeq \frac{8477}{16383}, \frac{7906}{16383}$$
$$\frac{3193}{16383}, \frac{3202}{16383} \simeq \frac{8491}{16383}, \frac{7892}{16383}$$
$$\frac{3194}{16383}, \frac{3201}{16383} \simeq \frac{7891}{16383}, \frac{8492}{16383}$$
$$\frac{3195}{16383}, \frac{3196}{16383} \simeq \frac{8497}{16383}, \frac{7886}{16383}$$
$$\frac{3197}{16383}, \frac{3198}{16383} \simeq \frac{2829}{5461}, \frac{2632}{5461}$$
$$\frac{3199}{16383}, \frac{3200}{16383} \simeq \frac{2631}{5461}, \frac{2830}{5461}$$
$$\frac{3203}{16383}, \frac{3204}{16383} \simeq \frac{8489}{16383}, \frac{7894}{16383}$$
$$\frac{3205}{16383}, \frac{3206}{16383} \simeq \frac{8495}{16383}, \frac{2832}{5461}$$

$$\frac{3207}{16383}, \frac{3208}{16383} \simeq \frac{2831}{5461}, \frac{2630}{5461}$$
$$\frac{3211}{16383}, \frac{3212}{16383} \simeq \frac{2635}{5461}, \frac{2826}{5461}$$
$$\frac{3213}{16383}, \frac{3214}{16383} \simeq \frac{7895}{16383}, \frac{8488}{16383}$$
$$\frac{3215}{16383}, \frac{3216}{16383} \simeq \frac{8485}{16383}, \frac{7898}{16383}$$
$$\frac{3219}{16383}, \frac{3220}{16383} \simeq \frac{3219}{5461}, \frac{9670}{16383}$$
$$\frac{3221}{16383}, \frac{3222}{16383} \simeq \frac{9664}{16383}, \frac{3221}{5461}$$
$$\frac{3223}{16383}, \frac{3224}{16383} \simeq \frac{9661}{16383}, \frac{3222}{5461}$$
$$\frac{3226}{16383}, \frac{3233}{16383} \simeq \frac{77}{129}, \frac{3260}{5461}$$
$$\frac{3227}{16383}, \frac{3228}{16383} \simeq \frac{3258}{5461}, \frac{3259}{5461}$$
$$\frac{3229}{16383}, \frac{3230}{16383} \simeq \frac{9799}{16383}, \frac{9800}{16383}$$
$$\frac{3231}{16383}, \frac{3232}{16383} \simeq \frac{9802}{16383}, \frac{9781}{16383}$$
$$\frac{3234}{16383}, \frac{4377}{16383} \simeq \frac{9676}{16383}, \frac{9803}{16383}$$
$$\frac{3235}{16383}, \frac{3236}{16383} \simeq \frac{9782}{16383}, \frac{3267}{5461}$$
$$\frac{3237}{16383}, \frac{3238}{16383} \simeq \frac{3237}{5461}, \frac{9712}{16383}$$
$$\frac{3239}{16383}, \frac{3240}{16383} \simeq \frac{3247}{5461}, \frac{9746}{16383}$$
$$\frac{3241}{16383}, \frac{3250}{16383} \simeq \frac{3241}{5461}, \frac{9724}{16383}$$
$$\frac{3242}{16383}, \frac{3249}{16383} \simeq \frac{9731}{16383}, \frac{3244}{5461}$$
$$\frac{3243}{16383}, \frac{3244}{16383} \simeq \frac{3243}{5461}, \frac{3242}{5461}$$

$$\frac{3245}{16383}, \frac{3246}{16383} \simeq \frac{3240}{5461}, \frac{9719}{16383}$$
$$\frac{3247}{16383}, \frac{3248}{16383} \simeq \frac{9722}{16383}, \frac{9733}{16383}$$
$$\frac{3251}{16383}, \frac{3252}{16383} \simeq \frac{3234}{5461}, \frac{3251}{5461}$$
$$\frac{3253}{16383}, \frac{3254}{16383} \simeq \frac{9695}{16383}, \frac{3232}{5461}$$
$$\frac{3255}{16383}, \frac{3256}{16383} \simeq \frac{3231}{5461}, \frac{9698}{16383}$$
$$\frac{3257}{16383}, \frac{3266}{16383} \simeq \frac{9707}{16383}, \frac{3236}{5461}$$
$$\frac{3258}{16383}, \frac{3265}{16383} \simeq \frac{3249}{5461}, \frac{9748}{16383}$$
$$\frac{3259}{16383}, \frac{3260}{16383} \simeq \frac{9742}{16383}, \frac{9745}{16383}$$
$$\frac{3261}{16383}, \frac{3262}{16383} \simeq \frac{9703}{16383}, \frac{9704}{16383}$$
$$\frac{3263}{16383}, \frac{3264}{16383} \simeq \frac{9749}{16383}, \frac{9706}{16383}$$
$$\frac{3267}{16383}, \frac{3268}{16383} \simeq \frac{3250}{5461}, \frac{3235}{5461}$$
$$\frac{3269}{16383}, \frac{3270}{16383} \simeq \frac{9743}{16383}, \frac{3248}{5461}$$
$$\frac{3271}{16383}, \frac{3272}{16383} \simeq \frac{3238}{5461}, \frac{9709}{16383}$$
$$\frac{3273}{16383}, \frac{4370}{16383} \simeq \frac{3273}{5461}, \frac{9820}{16383}$$
$$\frac{3274}{16383}, \frac{4369}{16383} \simeq \frac{9827}{16383}, \frac{3276}{5461}$$
$$\frac{3275}{16383}, \frac{3276}{16383} \simeq \frac{3274}{5461}, \frac{3275}{5461}$$
$$\frac{3277}{16383}, \frac{3278}{16383} \simeq \frac{6551}{16383}, \frac{2184}{5461}$$
$$\frac{3279}{16383}, \frac{3280}{16383} \simeq \frac{6565}{16383}, \frac{6554}{16383}$$

$$\frac{3281}{16383}, \frac{3338}{16383} \simeq \frac{6563}{16383}, \frac{2188}{5461}$$
$$\frac{3282}{16383}, \frac{3337}{16383} \simeq \frac{2185}{5461}, \frac{6556}{16383}$$
$$\frac{3283}{16383}, \frac{3300}{16383} \simeq \frac{6521}{16383}, \frac{6518}{16383}$$
$$\frac{3284}{16383}, \frac{3299}{16383} \simeq \frac{2178}{5461}, \frac{2179}{5461}$$
$$\frac{3285}{16383}, \frac{3286}{16383} \simeq \frac{6527}{16383}, \frac{2176}{5461}$$
$$\frac{3287}{16383}, \frac{3288}{16383} \simeq \frac{2175}{5461}, \frac{6530}{16383}$$
$$\frac{3289}{16383}, \frac{3298}{16383} \simeq \frac{6539}{16383}, \frac{2180}{5461}$$
$$\frac{3290}{16383}, \frac{3297}{16383} \simeq \frac{6515}{16383}, \frac{2172}{5461}$$
$$\frac{3291}{16383}, \frac{3292}{16383} \simeq \frac{2171}{5461}, \frac{2170}{5461}$$
$$\frac{3293}{16383}, \frac{3294}{16383} \simeq \frac{6535}{16383}, \frac{152}{381}$$
$$\frac{3295}{16383}, \frac{3296}{16383} \simeq \frac{6538}{16383}, \frac{6517}{16383}$$
$$\frac{3301}{16383}, \frac{3302}{16383} \simeq \frac{6575}{16383}, \frac{2192}{5461}$$
$$\frac{3303}{16383}, \frac{3304}{16383} \simeq \frac{2246}{5461}, \frac{6733}{16383}$$
$$\frac{3305}{16383}, \frac{3314}{16383} \simeq \frac{6715}{16383}, \frac{6716}{16383}$$
$$\frac{3306}{16383}, \frac{3313}{16383} \simeq \frac{2241}{5461}, \frac{6724}{16383}$$
$$\frac{3307}{16383}, \frac{3308}{16383} \simeq \frac{6718}{16383}, \frac{6721}{16383}$$
$$\frac{3309}{16383}, \frac{3310}{16383} \simeq \frac{2237}{5461}, \frac{6712}{16383}$$
$$\frac{3311}{16383}, \frac{3312}{16383} \simeq \frac{6725}{16383}, \frac{2238}{5461}$$
$$\frac{3315}{16383}, \frac{3332}{16383} \simeq \frac{6745}{16383}, \frac{6566}{16383}$$
$$\frac{3316}{16383}, \frac{3331}{16383} \simeq \frac{6742}{16383}, \frac{6569}{16383}$$
$$\frac{3317}{16383}, \frac{3318}{16383} \simeq \frac{6559}{16383}, \frac{6560}{16383}$$
$$\frac{3319}{16383}, \frac{3320}{16383} \simeq \frac{6562}{16383}, \frac{6557}{16383}$$
$$\frac{3321}{16383}, \frac{3330}{16383} \simeq \frac{6572}{16383}, \frac{6571}{16383}$$
$$\frac{3322}{16383}, \frac{3329}{16383} \simeq \frac{6740}{16383}, \frac{6739}{16383}$$
$$\frac{3323}{16383}, \frac{3324}{16383} \simeq \frac{6737}{16383}, \frac{6734}{16383}$$
$$\frac{3325}{16383}, \frac{3326}{16383} \simeq \frac{6568}{16383}, \frac{2189}{5461}$$
$$\frac{3327}{16383}, \frac{3328}{16383} \simeq \frac{2247}{5461}, \frac{2190}{5461}$$
$$\frac{3333}{16383}, \frac{3334}{16383} \simeq \frac{2245}{5461}, \frac{6736}{16383}$$
$$\frac{3335}{16383}, \frac{3336}{16383} \simeq \frac{6578}{16383}, \frac{2191}{5461}$$
$$\frac{3339}{16383}, \frac{3340}{16383} \simeq \frac{2187}{5461}, \frac{2186}{5461}$$
$$\frac{3341}{16383}, \frac{3342}{16383} \simeq \frac{6743}{16383}, \frac{2248}{5461}$$
$$\frac{3343}{16383}, \frac{3344}{16383} \simeq \frac{6757}{16383}, \frac{6746}{16383}$$
$$\frac{3345}{16383}, \frac{4298}{16383} \simeq \frac{6755}{16383}, \frac{2252}{5461}$$
$$\frac{3346}{16383}, \frac{4297}{16383} \simeq \frac{2249}{5461}, \frac{6748}{16383}$$
$$\frac{3347}{16383}, \frac{4260}{16383} \simeq \frac{6713}{16383}, \frac{2242}{5461}$$
$$\frac{3348}{16383}, \frac{4259}{16383} \simeq \frac{6710}{16383}, \frac{2243}{5461}$$

$\frac{3349}{16383}, \frac{3350}{16383}$	$\simeq$	$\frac{2240}{5461}, \frac{6719}{16383}$	$\frac{3380}{16383}, \frac{3651}{16383}$	$\simeq$	$\frac{5993}{16383}, \frac{5990}{16383}$
$\frac{3351}{16383}, \frac{3352}{16383}$	$\simeq$	$\frac{2239}{5461}, \frac{6722}{16383}$	$\frac{3381}{16383}, \frac{3382}{16383}$	$\simeq$	$\frac{5983}{16383}, \frac{5984}{16383}$
$\frac{3353}{16383}, \frac{4258}{16383}$	$\simeq$	$\frac{53}{129}, \frac{2244}{5461}$	$\frac{3383}{16383}, \frac{3384}{16383}$	$\simeq$	$\frac{2079}{5461}, \frac{5986}{16383}$
$\frac{3355}{16383}, \frac{3356}{16383}$	$\simeq$	$\frac{6577}{16383}, \frac{6574}{16383}$	$\frac{3385}{16383}, \frac{3394}{16383}$	$\simeq$	$\frac{6251}{16383}, \frac{6284}{16383}$
$\frac{3357}{16383}, \frac{3358}{16383}$	$\simeq$	$\frac{6599}{16383}, \frac{2200}{5461}$	$\frac{3386}{16383}, \frac{3393}{16383}$	$\simeq$	$\frac{2097}{5461}, \frac{6292}{16383}$
$\frac{3359}{16383}, \frac{3360}{16383}$	$\simeq$	$\frac{6602}{16383}, \frac{6581}{16383}$	$\frac{3387}{16383}, \frac{3388}{16383}$	$\simeq$	$\frac{6286}{16383}, \frac{6289}{16383}$
$\frac{3361}{16383}, \frac{4250}{16383}$	$\simeq$	$\frac{6580}{16383}, \frac{6707}{16383}$	$\frac{3389}{16383}, \frac{3390}{16383}$	$\simeq$	$\frac{6247}{16383}, \frac{6248}{16383}$
$\frac{3362}{16383}, \frac{4249}{16383}$	$\simeq$	$\frac{2201}{5461}, \frac{52}{129}$	$\frac{3391}{16383}, \frac{3392}{16383}$	$\simeq$	$\frac{6293}{16383}, \frac{6250}{16383}$
$\frac{3363}{16383}, \frac{4244}{16383}$	$\simeq$	$\frac{2194}{5461}, \frac{6601}{16383}$	$\frac{3395}{16383}, \frac{3636}{16383}$	$\simeq$	$\frac{2083}{5461}, \frac{2082}{5461}$
$\frac{3364}{16383}, \frac{4243}{16383}$	$\simeq$	$\frac{2195}{5461}, \frac{6598}{16383}$	$\frac{3396}{16383}, \frac{3635}{16383}$	$\simeq$	$\frac{2098}{5461}, \frac{139}{381}$
$\frac{3365}{16383}, \frac{3366}{16383}$	$\simeq$	$\frac{5999}{16383}, \frac{2000}{5461}$	$\frac{3397}{16383}, \frac{3398}{16383}$	$\simeq$	$\frac{6287}{16383}, \frac{2096}{5461}$
$\frac{3367}{16383}, \frac{3368}{16383}$	$\simeq$	$\frac{6029}{16383}, \frac{6034}{16383}$	$\frac{3399}{16383}, \frac{3400}{16383}$	$\simeq$	$\frac{2086}{5461}, \frac{6253}{16383}$
$\frac{3369}{16383}, \frac{3378}{16383}$	$\simeq$	$\frac{6011}{16383}, \frac{2004}{5461}$	$\frac{3401}{16383}, \frac{3474}{16383}$	$\simeq$	$\frac{6107}{16383}, \frac{2036}{5461}$
$\frac{3370}{16383}, \frac{3377}{16383}$	$\simeq$	$\frac{6019}{16383}, \frac{140}{381}$	$\frac{3402}{16383}, \frac{3473}{16383}$	$\simeq$	$\frac{6115}{16383}, \frac{6116}{16383}$
$\frac{3371}{16383}, \frac{3372}{16383}$	$\simeq$	$\frac{6014}{16383}, \frac{6017}{16383}$	$\frac{3403}{16383}, \frac{3404}{16383}$	$\simeq$	$\frac{6110}{16383}, \frac{6113}{16383}$
$\frac{3373}{16383}, \frac{3374}{16383}$	$\simeq$	$\frac{6007}{16383}, \frac{6008}{16383}$	$\frac{3405}{16383}, \frac{3470}{16383}$	$\simeq$	$\frac{6103}{16383}, \frac{2056}{5461}$
$\frac{3375}{16383}, \frac{3376}{16383}$	$\simeq$	$\frac{2007}{5461}, \frac{6010}{16383}$	$\frac{3406}{16383}, \frac{3469}{16383}$	$\simeq$	$\frac{6104}{16383}, \frac{6167}{16383}$
$\frac{3379}{16383}, \frac{3652}{16383}$	$\simeq$	$\frac{6233}{16383}, \frac{6038}{16383}$	$\frac{3407}{16383}, \frac{3408}{16383}$	$\simeq$	$\frac{6170}{16383}, \frac{6181}{16383}$

$$\frac{3409}{16383},\frac{3466}{16383} \simeq \frac{6179}{16383},\frac{2060}{5461}$$
$$\frac{3410}{16383},\frac{3465}{16383} \simeq \frac{2057}{5461},\frac{6172}{16383}$$
$$\frac{3411}{16383},\frac{3428}{16383} \simeq \frac{6134}{16383},\frac{6137}{16383}$$
$$\frac{3412}{16383},\frac{3427}{16383} \simeq \frac{2050}{5461},\frac{2051}{5461}$$
$$\frac{3413}{16383},\frac{3414}{16383} \simeq \frac{2048}{5461},\frac{6143}{16383}$$
$$\frac{3415}{16383},\frac{3416}{16383} \simeq \frac{2047}{5461},\frac{6146}{16383}$$
$$\frac{3417}{16383},\frac{3426}{16383} \simeq \frac{6124}{16383},\frac{6155}{16383}$$
$$\frac{3418}{16383},\frac{3425}{16383} \simeq \frac{6131}{16383},\frac{2044}{5461}$$
$$\frac{3419}{16383},\frac{3420}{16383} \simeq \frac{2042}{5461},\frac{2043}{5461}$$
$$\frac{3421}{16383},\frac{3422}{16383} \simeq \frac{6151}{16383},\frac{6152}{16383}$$
$$\frac{3423}{16383},\frac{3424}{16383} \simeq \frac{6133}{16383},\frac{6154}{16383}$$
$$\frac{3429}{16383},\frac{3430}{16383} \simeq \frac{6191}{16383},\frac{6064}{16383}$$
$$\frac{3431}{16383},\frac{3432}{16383} \simeq \frac{6098}{16383},\frac{2031}{5461}$$
$$\frac{3433}{16383},\frac{3442}{16383} \simeq \frac{2025}{5461},\frac{6076}{16383}$$
$$\frac{3434}{16383},\frac{3441}{16383} \simeq \frac{6083}{16383},\frac{2028}{5461}$$
$$\frac{3435}{16383},\frac{3436}{16383} \simeq \frac{2026}{5461},\frac{2027}{5461}$$
$$\frac{3437}{16383},\frac{3438}{16383} \simeq \frac{6071}{16383},\frac{2024}{5461}$$
$$\frac{3439}{16383},\frac{3440}{16383} \simeq \frac{6085}{16383},\frac{6074}{16383}$$

$$\frac{3443}{16383},\frac{3460}{16383} \simeq \frac{6169}{16383},\frac{2034}{5461}$$
$$\frac{3444}{16383},\frac{3459}{16383} \simeq \frac{6182}{16383},\frac{6185}{16383}$$
$$\frac{3445}{16383},\frac{3446}{16383} \simeq \frac{6175}{16383},\frac{6176}{16383}$$
$$\frac{3447}{16383},\frac{3448}{16383} \simeq \frac{6173}{16383},\frac{6178}{16383}$$
$$\frac{3449}{16383},\frac{3458}{16383} \simeq \frac{6187}{16383},\frac{6092}{16383}$$
$$\frac{3450}{16383},\frac{3457}{16383} \simeq \frac{2033}{5461},\frac{6100}{16383}$$
$$\frac{3451}{16383},\frac{3452}{16383} \simeq \frac{6097}{16383},\frac{6094}{16383}$$
$$\frac{3453}{16383},\frac{3454}{16383} \simeq \frac{2061}{5461},\frac{6184}{16383}$$
$$\frac{3455}{16383},\frac{3456}{16383} \simeq \frac{2062}{5461},\frac{6101}{16383}$$
$$\frac{3461}{16383},\frac{3462}{16383} \simeq \frac{6095}{16383},\frac{16}{43}$$
$$\frac{3463}{16383},\frac{3464}{16383} \simeq \frac{2063}{5461},\frac{2022}{5461}$$
$$\frac{3467}{16383},\frac{3468}{16383} \simeq \frac{2058}{5461},\frac{2059}{5461}$$
$$\frac{3471}{16383},\frac{3472}{16383} \simeq \frac{2039}{5461},\frac{142}{381}$$
$$\frac{3475}{16383},\frac{3620}{16383} \simeq \frac{6329}{16383},\frac{6326}{16383}$$
$$\frac{3476}{16383},\frac{3619}{16383} \simeq \frac{5833}{16383},\frac{5830}{16383}$$
$$\frac{3477}{16383},\frac{3478}{16383} \simeq \frac{1941}{5461},\frac{5824}{16383}$$
$$\frac{3479}{16383},\frac{3480}{16383} \simeq \frac{1942}{5461},\frac{5821}{16383}$$
$$\frac{3481}{16383},\frac{3618}{16383} \simeq \frac{5932}{16383},\frac{1945}{5461}$$

$$\frac{3482}{16383}, \frac{3617}{16383} \simeq \frac{6323}{16383}, \frac{2108}{5461}$$
$$\frac{3484}{16383}, \frac{3611}{16383} \simeq \frac{1979}{5461}, \frac{2106}{5461}$$
$$\frac{3485}{16383}, \frac{3486}{16383} \simeq \frac{5959}{16383}, \frac{5960}{16383}$$
$$\frac{3487}{16383}, \frac{3488}{16383} \simeq \frac{5962}{16383}, \frac{5941}{16383}$$
$$\frac{3489}{16383}, \frac{3610}{16383} \simeq \frac{1980}{5461}, \frac{5939}{16383}$$
$$\frac{3490}{16383}, \frac{3609}{16383} \simeq \frac{5963}{16383}, \frac{6316}{16383}$$
$$\frac{3491}{16383}, \frac{3604}{16383} \simeq \frac{1986}{5461}, \frac{1987}{5461}$$
$$\frac{3492}{16383}, \frac{3603}{16383} \simeq \frac{5945}{16383}, \frac{5942}{16383}$$
$$\frac{3493}{16383}, \frac{3494}{16383} \simeq \frac{1957}{5461}, \frac{5872}{16383}$$
$$\frac{3495}{16383}, \frac{3496}{16383} \simeq \frac{1967}{5461}, \frac{5906}{16383}$$
$$\frac{3497}{16383}, \frac{3506}{16383} \simeq \frac{1961}{5461}, \frac{5884}{16383}$$
$$\frac{3498}{16383}, \frac{3505}{16383} \simeq \frac{137}{381}, \frac{1964}{5461}$$
$$\frac{3499}{16383}, \frac{3500}{16383} \simeq \frac{1962}{5461}, \frac{1963}{5461}$$
$$\frac{3501}{16383}, \frac{3502}{16383} \simeq \frac{5879}{16383}, \frac{1960}{5461}$$
$$\frac{3503}{16383}, \frac{3504}{16383} \simeq \frac{5893}{16383}, \frac{5882}{16383}$$
$$\frac{3507}{16383}, \frac{3524}{16383} \simeq \frac{1970}{5461}, \frac{5849}{16383}$$
$$\frac{3508}{16383}, \frac{3523}{16383} \simeq \frac{1955}{5461}, \frac{1954}{5461}$$
$$\frac{3509}{16383}, \frac{3510}{16383} \simeq \frac{5855}{16383}, \frac{1952}{5461}$$
$$\frac{3511}{16383}, \frac{3512}{16383} \simeq \frac{5858}{16383}, \frac{1951}{5461}$$
$$\frac{3513}{16383}, \frac{3522}{16383} \simeq \frac{5900}{16383}, \frac{5867}{16383}$$
$$\frac{3514}{16383}, \frac{3521}{16383} \simeq \frac{1969}{5461}, \frac{5908}{16383}$$
$$\frac{3515}{16383}, \frac{3516}{16383} \simeq \frac{5905}{16383}, \frac{5902}{16383}$$
$$\frac{3517}{16383}, \frac{3518}{16383} \simeq \frac{5864}{16383}, \frac{5863}{16383}$$
$$\frac{3519}{16383}, \frac{3520}{16383} \simeq \frac{5909}{16383}, \frac{5866}{16383}$$
$$\frac{3525}{16383}, \frac{3526}{16383} \simeq \frac{5903}{16383}, \frac{1968}{5461}$$
$$\frac{3527}{16383}, \frac{3528}{16383} \simeq \frac{1958}{5461}, \frac{5869}{16383}$$
$$\frac{3529}{16383}, \frac{3602}{16383} \simeq \frac{5980}{16383}, \frac{1993}{5461}$$
$$\frac{3530}{16383}, \frac{3601}{16383} \simeq \frac{2081}{5461}, \frac{6244}{16383}$$
$$\frac{3531}{16383}, \frac{3532}{16383} \simeq \frac{6238}{16383}, \frac{6241}{16383}$$
$$\frac{3533}{16383}, \frac{3598}{16383} \simeq \frac{1992}{5461}, \frac{5975}{16383}$$
$$\frac{3534}{16383}, \frac{3597}{16383} \simeq \frac{6295}{16383}, \frac{6296}{16383}$$
$$\frac{3535}{16383}, \frac{3536}{16383} \simeq \frac{2103}{5461}, \frac{6298}{16383}$$
$$\frac{3537}{16383}, \frac{3594}{16383} \simeq \frac{6307}{16383}, \frac{6308}{16383}$$
$$\frac{3538}{16383}, \frac{3593}{16383} \simeq \frac{6299}{16383}, \frac{2100}{5461}$$
$$\frac{3539}{16383}, \frac{3556}{16383} \simeq \frac{6265}{16383}, \frac{6262}{16383}$$
$$\frac{3540}{16383}, \frac{3555}{16383} \simeq \frac{146}{381}, \frac{6281}{16383}$$

$$\frac{3541}{16383}, \frac{3542}{16383} \simeq \frac{6271}{16383}, \frac{6272}{16383}$$
$$\frac{3543}{16383}, \frac{3544}{16383} \simeq \frac{6269}{16383}, \frac{6274}{16383}$$
$$\frac{3545}{16383}, \frac{3554}{16383} \simeq \frac{6283}{16383}, \frac{2084}{5461}$$
$$\frac{3546}{16383}, \frac{3553}{16383} \simeq \frac{6259}{16383}, \frac{6260}{16383}$$
$$\frac{3547}{16383}, \frac{3548}{16383} \simeq \frac{6257}{16383}, \frac{6254}{16383}$$
$$\frac{3549}{16383}, \frac{3550}{16383} \simeq \frac{2093}{5461}, \frac{6280}{16383}$$
$$\frac{3551}{16383}, \frac{3552}{16383} \simeq \frac{2094}{5461}, \frac{2087}{5461}$$
$$\frac{3557}{16383}, \frac{3558}{16383} \simeq \frac{5936}{16383}, \frac{5935}{16383}$$
$$\frac{3559}{16383}, \frac{3560}{16383} \simeq \frac{1990}{5461}, \frac{5965}{16383}$$
$$\frac{3561}{16383}, \frac{3570}{16383} \simeq \frac{5948}{16383}, \frac{5947}{16383}$$
$$\frac{3562}{16383}, \frac{3569}{16383} \simeq \frac{1985}{5461}, \frac{5956}{16383}$$
$$\frac{3563}{16383}, \frac{3564}{16383} \simeq \frac{5953}{16383}, \frac{5950}{16383}$$
$$\frac{3565}{16383}, \frac{3566}{16383} \simeq \frac{5944}{16383}, \frac{1981}{5461}$$
$$\frac{3567}{16383}, \frac{3568}{16383} \simeq \frac{5957}{16383}, \frac{1982}{5461}$$
$$\frac{3571}{16383}, \frac{3588}{16383} \simeq \frac{2099}{5461}, \frac{5974}{16383}$$
$$\frac{3572}{16383}, \frac{3587}{16383} \simeq \frac{6313}{16383}, \frac{6310}{16383}$$
$$\frac{3573}{16383}, \frac{3574}{16383} \simeq \frac{2101}{5461}, \frac{6304}{16383}$$
$$\frac{3575}{16383}, \frac{3576}{16383} \simeq \frac{2102}{5461}, \frac{6301}{16383}$$
$$\frac{3577}{16383}, \frac{3586}{16383} \simeq \frac{2105}{5461}, \frac{1988}{5461}$$
$$\frac{3578}{16383}, \frac{3585}{16383} \simeq \frac{5972}{16383}, \frac{5971}{16383}$$
$$\frac{3579}{16383}, \frac{3580}{16383} \simeq \frac{47}{129}, \frac{5966}{16383}$$
$$\frac{3581}{16383}, \frac{3582}{16383} \simeq \frac{2104}{5461}, \frac{6311}{16383}$$
$$\frac{3583}{16383}, \frac{3584}{16383} \simeq \frac{6314}{16383}, \frac{1991}{5461}$$
$$\frac{3589}{16383}, \frac{3590}{16383} \simeq \frac{5968}{16383}, \frac{1989}{5461}$$
$$\frac{3591}{16383}, \frac{3592}{16383} \simeq \frac{6317}{16383}, \frac{5938}{16383}$$
$$\frac{3595}{16383}, \frac{3596}{16383} \simeq \frac{6302}{16383}, \frac{6305}{16383}$$
$$\frac{3599}{16383}, \frac{3600}{16383} \simeq \frac{6245}{16383}, \frac{5978}{16383}$$
$$\frac{3605}{16383}, \frac{3606}{16383} \simeq \frac{5951}{16383}, \frac{1984}{5461}$$
$$\frac{3607}{16383}, \frac{3608}{16383} \simeq \frac{5954}{16383}, \frac{1983}{5461}$$
$$\frac{3613}{16383}, \frac{3614}{16383} \simeq \frac{1944}{5461}, \frac{5831}{16383}$$
$$\frac{3615}{16383}, \frac{3616}{16383} \simeq \frac{5834}{16383}, \frac{6325}{16383}$$
$$\frac{3621}{16383}, \frac{3622}{16383} \simeq \frac{2085}{5461}, \frac{6256}{16383}$$
$$\frac{3623}{16383}, \frac{3624}{16383} \simeq \frac{2095}{5461}, \frac{6290}{16383}$$
$$\frac{3625}{16383}, \frac{3634}{16383} \simeq \frac{2089}{5461}, \frac{6268}{16383}$$
$$\frac{3626}{16383}, \frac{3633}{16383} \simeq \frac{6275}{16383}, \frac{2092}{5461}$$
$$\frac{3627}{16383}, \frac{3628}{16383} \simeq \frac{2090}{5461}, \frac{2091}{5461}$$

$$\frac{3629}{16383},\frac{3630}{16383} \simeq \frac{6263}{16383},\frac{2088}{5461}$$
$$\frac{3631}{16383},\frac{3632}{16383} \simeq \frac{6277}{16383},\frac{6266}{16383}$$
$$\frac{3637}{16383},\frac{3638}{16383} \simeq \frac{6239}{16383},\frac{2080}{5461}$$
$$\frac{3639}{16383},\frac{3640}{16383} \simeq \frac{6242}{16383},\frac{5981}{16383}$$
$$\frac{3641}{16383},\frac{3650}{16383} \simeq \frac{6028}{16383},\frac{5995}{16383}$$
$$\frac{3642}{16383},\frac{3649}{16383} \simeq \frac{2012}{5461},\frac{6035}{16383}$$
$$\frac{3643}{16383},\frac{3644}{16383} \simeq \frac{2011}{5461},\frac{2010}{5461}$$
$$\frac{3645}{16383},\frac{3646}{16383} \simeq \frac{5992}{16383},\frac{1997}{5461}$$
$$\frac{3647}{16383},\frac{3648}{16383} \simeq \frac{1998}{5461},\frac{6037}{16383}$$
$$\frac{3653}{16383},\frac{3654}{16383} \simeq \frac{6031}{16383},\frac{6032}{16383}$$
$$\frac{3655}{16383},\frac{3656}{16383} \simeq \frac{6002}{16383},\frac{1999}{5461}$$
$$\frac{3657}{16383},\frac{4242}{16383} \simeq \frac{2292}{5461},\frac{6875}{16383}$$
$$\frac{3658}{16383},\frac{4241}{16383} \simeq \frac{6883}{16383},\frac{6884}{16383}$$
$$\frac{3659}{16383},\frac{3660}{16383} \simeq \frac{6878}{16383},\frac{6881}{16383}$$
$$\frac{3661}{16383},\frac{3662}{16383} \simeq \frac{6935}{16383},\frac{2312}{5461}$$
$$\frac{3663}{16383},\frac{3664}{16383} \simeq \frac{6949}{16383},\frac{6874}{16383}$$
$$\frac{3665}{16383},\frac{4234}{16383} \simeq \frac{6947}{16383},\frac{2316}{5461}$$
$$\frac{3666}{16383},\frac{4233}{16383} \simeq \frac{2313}{5461},\frac{6940}{16383}$$

$$\frac{3667}{16383},\frac{3684}{16383} \simeq \frac{6905}{16383},\frac{2306}{5461}$$
$$\frac{3668}{16383},\frac{3683}{16383} \simeq \frac{6902}{16383},\frac{2307}{5461}$$
$$\frac{3669}{16383},\frac{3670}{16383} \simeq \frac{2304}{5461},\frac{6911}{16383}$$
$$\frac{3671}{16383},\frac{3672}{16383} \simeq \frac{2303}{5461},\frac{6914}{16383}$$
$$\frac{3673}{16383},\frac{3682}{16383} \simeq \frac{161}{381},\frac{2308}{5461}$$
$$\frac{3674}{16383},\frac{3681}{16383} \simeq \frac{6899}{16383},\frac{2300}{5461}$$
$$\frac{3675}{16383},\frac{3676}{16383} \simeq \frac{2298}{5461},\frac{2299}{5461}$$
$$\frac{3677}{16383},\frac{3678}{16383} \simeq \frac{6919}{16383},\frac{6920}{16383}$$
$$\frac{3679}{16383},\frac{3680}{16383} \simeq \frac{6922}{16383},\frac{6901}{16383}$$
$$\frac{3685}{16383},\frac{3686}{16383} \simeq \frac{6959}{16383},\frac{2320}{5461}$$
$$\frac{3687}{16383},\frac{3688}{16383} \simeq \frac{3142}{5461},\frac{2319}{5461}$$
$$\frac{3689}{16383},\frac{3698}{16383} \simeq \frac{9403}{16383},\frac{9404}{16383}$$
$$\frac{3690}{16383},\frac{3697}{16383} \simeq \frac{3137}{5461},\frac{9412}{16383}$$
$$\frac{3691}{16383},\frac{3692}{16383} \simeq \frac{9406}{16383},\frac{9409}{16383}$$
$$\frac{3693}{16383},\frac{3694}{16383} \simeq \frac{3133}{5461},\frac{9400}{16383}$$
$$\frac{3695}{16383},\frac{3696}{16383} \simeq \frac{9413}{16383},\frac{3134}{5461}$$
$$\frac{3699}{16383},\frac{4228}{16383} \simeq \frac{3170}{5461},\frac{9433}{16383}$$
$$\frac{3700}{16383},\frac{3715}{16383} \simeq \frac{3171}{5461},\frac{2290}{5461}$$

$$\frac{3701}{16383}, \frac{3702}{16383} \simeq \frac{221}{381}, \frac{3168}{5461}$$
$$\frac{3703}{16383}, \frac{3704}{16383} \simeq \frac{9506}{16383}, \frac{3167}{5461}$$
$$\frac{3705}{16383}, \frac{3714}{16383} \simeq \frac{3172}{5461}, \frac{9515}{16383}$$
$$\frac{3706}{16383}, \frac{3713}{16383} \simeq \frac{6868}{16383}, \frac{2289}{5461}$$
$$\frac{3707}{16383}, \frac{3708}{16383} \simeq \frac{6865}{16383}, \frac{6862}{16383}$$
$$\frac{3709}{16383}, \frac{3710}{16383} \simeq \frac{9512}{16383}, \frac{9511}{16383}$$
$$\frac{3711}{16383}, \frac{3712}{16383} \simeq \frac{9514}{16383}, \frac{6869}{16383}$$
$$\frac{3716}{16383}, \frac{4211}{16383} \simeq \frac{6950}{16383}, \frac{2291}{5461}$$
$$\frac{3717}{16383}, \frac{3718}{16383} \simeq \frac{6863}{16383}, \frac{2288}{5461}$$
$$\frac{3719}{16383}, \frac{4200}{16383} \simeq \frac{3174}{5461}, \frac{9517}{16383}$$
$$\frac{3720}{16383}, \frac{4199}{16383} \simeq \frac{6866}{16383}, \frac{2287}{5461}$$
$$\frac{3721}{16383}, \frac{4178}{16383} \simeq \frac{9499}{16383}, \frac{9500}{16383}$$
$$\frac{3722}{16383}, \frac{4177}{16383} \simeq \frac{3169}{5461}, \frac{9508}{16383}$$
$$\frac{3723}{16383}, \frac{3724}{16383} \simeq \frac{9502}{16383}, \frac{9505}{16383}$$
$$\frac{3725}{16383}, \frac{3726}{16383} \simeq \frac{9431}{16383}, \frac{3144}{5461}$$
$$\frac{3727}{16383}, \frac{3728}{16383} \simeq \frac{3166}{5461}, \frac{9445}{16383}$$
$$\frac{3729}{16383}, \frac{4170}{16383} \simeq \frac{9443}{16383}, \frac{3148}{5461}$$
$$\frac{3730}{16383}, \frac{4169}{16383} \simeq \frac{3145}{5461}, \frac{9436}{16383}$$

$$\frac{3731}{16383}, \frac{3876}{16383} \simeq \frac{9145}{16383}, \frac{9158}{16383}$$
$$\frac{3732}{16383}, \frac{3875}{16383} \simeq \frac{9142}{16383}, \frac{9289}{16383}$$
$$\frac{3733}{16383}, \frac{3734}{16383} \simeq \frac{9152}{16383}, \frac{9151}{16383}$$
$$\frac{3735}{16383}, \frac{3736}{16383} \simeq \frac{9149}{16383}, \frac{9154}{16383}$$
$$\frac{3737}{16383}, \frac{3874}{16383} \simeq \frac{9163}{16383}, \frac{9268}{16383}$$
$$\frac{3738}{16383}, \frac{3873}{16383} \simeq \frac{9164}{16383}, \frac{3089}{5461}$$
$$\frac{3739}{16383}, \frac{3740}{16383} \simeq \frac{9262}{16383}, \frac{9265}{16383}$$
$$\frac{3742}{16383}, \frac{3869}{16383} \simeq \frac{9160}{16383}, \frac{9287}{16383}$$
$$\frac{3743}{16383}, \frac{3744}{16383} \simeq \frac{9290}{16383}, \frac{3047}{5461}$$
$$\frac{3745}{16383}, \frac{3866}{16383} \simeq \frac{9139}{16383}, \frac{9292}{16383}$$
$$\frac{3746}{16383}, \frac{3865}{16383} \simeq \frac{9140}{16383}, \frac{3097}{5461}$$
$$\frac{3747}{16383}, \frac{3860}{16383} \simeq \frac{9161}{16383}, \frac{3090}{5461}$$
$$\frac{3748}{16383}, \frac{3859}{16383} \simeq \frac{3091}{5461}, \frac{9286}{16383}$$
$$\frac{3749}{16383}, \frac{3750}{16383} \simeq \frac{9199}{16383}, \frac{9200}{16383}$$
$$\frac{3751}{16383}, \frac{3784}{16383} \simeq \frac{214}{381}, \frac{9229}{16383}$$
$$\frac{3752}{16383}, \frac{3783}{16383} \simeq \frac{9197}{16383}, \frac{3078}{5461}$$
$$\frac{3753}{16383}, \frac{3762}{16383} \simeq \frac{9211}{16383}, \frac{9220}{16383}$$
$$\frac{3754}{16383}, \frac{3761}{16383} \simeq \frac{9212}{16383}, \frac{3073}{5461}$$

$$\frac{3755}{16383}, \frac{3756}{16383} \simeq \frac{9214}{16383}, \frac{9217}{16383}$$
$$\frac{3757}{16383}, \frac{3758}{16383} \simeq \frac{3069}{5461}, \frac{9208}{16383}$$
$$\frac{3759}{16383}, \frac{3760}{16383} \simeq \frac{3070}{5461}, \frac{9221}{16383}$$
$$\frac{3763}{16383}, \frac{3780}{16383} \simeq \frac{9190}{16383}, \frac{9241}{16383}$$
$$\frac{3764}{16383}, \frac{3779}{16383} \simeq \frac{9193}{16383}, \frac{9238}{16383}$$
$$\frac{3765}{16383}, \frac{3766}{16383} \simeq \frac{3061}{5461}, \frac{9184}{16383}$$
$$\frac{3767}{16383}, \frac{3768}{16383} \simeq \frac{9181}{16383}, \frac{3062}{5461}$$
$$\frac{3769}{16383}, \frac{3778}{16383} \simeq \frac{3065}{5461}, \frac{9236}{16383}$$
$$\frac{3770}{16383}, \frac{3777}{16383} \simeq \frac{9196}{16383}, \frac{9235}{16383}$$
$$\frac{3771}{16383}, \frac{3772}{16383} \simeq \frac{9230}{16383}, \frac{9233}{16383}$$
$$\frac{3773}{16383}, \frac{3774}{16383} \simeq \frac{9191}{16383}, \frac{3064}{5461}$$
$$\frac{3775}{16383}, \frac{3776}{16383} \simeq \frac{3079}{5461}, \frac{9194}{16383}$$
$$\frac{3781}{16383}, \frac{3782}{16383} \simeq \frac{3077}{5461}, \frac{9232}{16383}$$
$$\frac{3785}{16383}, \frac{3858}{16383} \simeq \frac{9307}{16383}, \frac{9124}{16383}$$
$$\frac{3786}{16383}, \frac{3857}{16383} \simeq \frac{9308}{16383}, \frac{3105}{5461}$$
$$\frac{3787}{16383}, \frac{3788}{16383} \simeq \frac{9310}{16383}, \frac{9313}{16383}$$
$$\frac{3789}{16383}, \frac{3790}{16383} \simeq \frac{3037}{5461}, \frac{9112}{16383}$$
$$\frac{3791}{16383}, \frac{3792}{16383} \simeq \frac{3102}{5461}, \frac{9125}{16383}$$
$$\frac{3793}{16383}, \frac{3850}{16383} \simeq \frac{3041}{5461}, \frac{212}{381}$$
$$\frac{3794}{16383}, \frac{3849}{16383} \simeq \frac{9115}{16383}, \frac{9316}{16383}$$
$$\frac{3795}{16383}, \frac{3812}{16383} \simeq \frac{3027}{5461}, \frac{9094}{16383}$$
$$\frac{3796}{16383}, \frac{3811}{16383} \simeq \frac{3026}{5461}, \frac{9097}{16383}$$
$$\frac{3797}{16383}, \frac{3798}{16383} \simeq \frac{3029}{5461}, \frac{9088}{16383}$$
$$\frac{3799}{16383}, \frac{3800}{16383} \simeq \frac{9085}{16383}, \frac{3030}{5461}$$
$$\frac{3801}{16383}, \frac{3810}{16383} \simeq \frac{3033}{5461}, \frac{9076}{16383}$$
$$\frac{3802}{16383}, \frac{3809}{16383} \simeq \frac{3025}{5461}, \frac{9100}{16383}$$
$$\frac{3803}{16383}, \frac{3804}{16383} \simeq \frac{211}{381}, \frac{9070}{16383}$$
$$\frac{3805}{16383}, \frac{3806}{16383} \simeq \frac{9095}{16383}, \frac{3032}{5461}$$
$$\frac{3807}{16383}, \frac{3808}{16383} \simeq \frac{9098}{16383}, \frac{9077}{16383}$$
$$\frac{3813}{16383}, \frac{3814}{16383} \simeq \frac{3045}{5461}, \frac{9136}{16383}$$
$$\frac{3815}{16383}, \frac{3848}{16383} \simeq \frac{9293}{16383}, \frac{3046}{5461}$$
$$\frac{3816}{16383}, \frac{3847}{16383} \simeq \frac{9298}{16383}, \frac{9133}{16383}$$
$$\frac{3817}{16383}, \frac{3826}{16383} \simeq \frac{9275}{16383}, \frac{9284}{16383}$$
$$\frac{3818}{16383}, \frac{3825}{16383} \simeq \frac{3092}{5461}, \frac{9283}{16383}$$
$$\frac{3819}{16383}, \frac{3820}{16383} \simeq \frac{9278}{16383}, \frac{9281}{16383}$$
$$\frac{3821}{16383}, \frac{3822}{16383} \simeq \frac{73}{129}, \frac{9272}{16383}$$

$$\frac{3823}{16383}, \frac{3824}{16383} \simeq \frac{3095}{5461}, \frac{9274}{16383}$$
$$\frac{3827}{16383}, \frac{3844}{16383} \simeq \frac{9305}{16383}, \frac{3042}{5461}$$
$$\frac{3828}{16383}, \frac{3843}{16383} \simeq \frac{9302}{16383}, \frac{3043}{5461}$$
$$\frac{3829}{16383}, \frac{3830}{16383} \simeq \frac{9119}{16383}, \frac{3040}{5461}$$
$$\frac{3831}{16383}, \frac{3832}{16383} \simeq \frac{9122}{16383}, \frac{3039}{5461}$$
$$\frac{3833}{16383}, \frac{3842}{16383} \simeq \frac{3100}{5461}, \frac{9131}{16383}$$
$$\frac{3834}{16383}, \frac{3841}{16383} \simeq \frac{9299}{16383}, \frac{3044}{5461}$$
$$\frac{3835}{16383}, \frac{3836}{16383} \simeq \frac{3099}{5461}, \frac{3098}{5461}$$
$$\frac{3837}{16383}, \frac{3838}{16383} \simeq \frac{9128}{16383}, \frac{9127}{16383}$$
$$\frac{3839}{16383}, \frac{3840}{16383} \simeq \frac{9301}{16383}, \frac{9130}{16383}$$
$$\frac{3845}{16383}, \frac{3846}{16383} \simeq \frac{9295}{16383}, \frac{9296}{16383}$$
$$\frac{3851}{16383}, \frac{3852}{16383} \simeq \frac{9121}{16383}, \frac{9118}{16383}$$
$$\frac{3853}{16383}, \frac{3854}{16383} \simeq \frac{3101}{5461}, \frac{9304}{16383}$$
$$\frac{3855}{16383}, \frac{3856}{16383} \simeq \frac{9317}{16383}, \frac{3038}{5461}$$
$$\frac{3861}{16383}, \frac{3862}{16383} \simeq \frac{3093}{5461}, \frac{9280}{16383}$$
$$\frac{3863}{16383}, \frac{3864}{16383} \simeq \frac{9277}{16383}, \frac{3094}{5461}$$
$$\frac{3867}{16383}, \frac{3868}{16383} \simeq \frac{9137}{16383}, \frac{9134}{16383}$$
$$\frac{3871}{16383}, \frac{3872}{16383} \simeq \frac{9269}{16383}, \frac{3054}{5461}$$

$$\frac{3877}{16383}, \frac{3878}{16383} \simeq \frac{9583}{16383}, \frac{9584}{16383}$$
$$\frac{3879}{16383}, \frac{4168}{16383} \simeq \frac{3206}{5461}, \frac{9613}{16383}$$
$$\frac{3880}{16383}, \frac{4167}{16383} \simeq \frac{9586}{16383}, \frac{9581}{16383}$$
$$\frac{3881}{16383}, \frac{3890}{16383} \simeq \frac{9595}{16383}, \frac{9596}{16383}$$
$$\frac{3882}{16383}, \frac{3889}{16383} \simeq \frac{3201}{5461}, \frac{9604}{16383}$$
$$\frac{3883}{16383}, \frac{3884}{16383} \simeq \frac{9598}{16383}, \frac{9601}{16383}$$
$$\frac{3885}{16383}, \frac{3886}{16383} \simeq \frac{3197}{5461}, \frac{9592}{16383}$$
$$\frac{3887}{16383}, \frac{3888}{16383} \simeq \frac{9605}{16383}, \frac{3198}{5461}$$
$$\frac{3891}{16383}, \frac{4164}{16383} \simeq \frac{9625}{16383}, \frac{6758}{16383}$$
$$\frac{3892}{16383}, \frac{4163}{16383} \simeq \frac{9622}{16383}, \frac{6761}{16383}$$
$$\frac{3893}{16383}, \frac{3894}{16383} \simeq \frac{157}{381}, \frac{6752}{16383}$$
$$\frac{3895}{16383}, \frac{3896}{16383} \simeq \frac{6754}{16383}, \frac{6749}{16383}$$
$$\frac{3897}{16383}, \frac{4162}{16383} \simeq \frac{6764}{16383}, \frac{6763}{16383}$$
$$\frac{3898}{16383}, \frac{4161}{16383} \simeq \frac{2268}{5461}, \frac{6803}{16383}$$
$$\frac{3899}{16383}, \frac{3900}{16383} \simeq \frac{2267}{5461}, \frac{2266}{5461}$$
$$\frac{3901}{16383}, \frac{3902}{16383} \simeq \frac{3192}{5461}, \frac{9575}{16383}$$
$$\frac{3903}{16383}, \frac{3904}{16383} \simeq \frac{3207}{5461}, \frac{9578}{16383}$$
$$\frac{3905}{16383}, \frac{4154}{16383} \simeq \frac{9620}{16383}, \frac{9619}{16383}$$

$$\frac{3906}{16383}, \frac{4153}{16383} \simeq \frac{9580}{16383}, \frac{3193}{5461}$$
$$\frac{3907}{16383}, \frac{4148}{16383} \simeq \frac{9577}{16383}, \frac{6806}{16383}$$
$$\frac{3908}{16383}, \frac{4147}{16383} \simeq \frac{9574}{16383}, \frac{6809}{16383}$$
$$\frac{3909}{16383}, \frac{3910}{16383} \simeq \frac{6799}{16383}, \frac{6800}{16383}$$
$$\frac{3911}{16383}, \frac{4136}{16383} \simeq \frac{6770}{16383}, \frac{2255}{5461}$$
$$\frac{3912}{16383}, \frac{4135}{16383} \simeq \frac{6802}{16383}, \frac{6797}{16383}$$
$$\frac{3913}{16383}, \frac{3986}{16383} \simeq \frac{6620}{16383}, \frac{6619}{16383}$$
$$\frac{3914}{16383}, \frac{3985}{16383} \simeq \frac{2209}{5461}, \frac{6628}{16383}$$
$$\frac{3915}{16383}, \frac{3916}{16383} \simeq \frac{154}{381}, \frac{6625}{16383}$$
$$\frac{3917}{16383}, \frac{3918}{16383} \simeq \frac{6679}{16383}, \frac{6680}{16383}$$
$$\frac{3919}{16383}, \frac{3984}{16383} \simeq \frac{2231}{5461}, \frac{6682}{16383}$$
$$\frac{3920}{16383}, \frac{3983}{16383} \simeq \frac{6629}{16383}, \frac{2206}{5461}$$
$$\frac{3921}{16383}, \frac{3978}{16383} \simeq \frac{6691}{16383}, \frac{6692}{16383}$$
$$\frac{3922}{16383}, \frac{3977}{16383} \simeq \frac{6683}{16383}, \frac{2228}{5461}$$
$$\frac{3923}{16383}, \frac{3940}{16383} \simeq \frac{6649}{16383}, \frac{6662}{16383}$$
$$\frac{3924}{16383}, \frac{3939}{16383} \simeq \frac{6646}{16383}, \frac{155}{381}$$
$$\frac{3925}{16383}, \frac{3926}{16383} \simeq \frac{6655}{16383}, \frac{6656}{16383}$$
$$\frac{3927}{16383}, \frac{3928}{16383} \simeq \frac{6653}{16383}, \frac{6658}{16383}$$

$$\frac{3929}{16383}, \frac{3938}{16383} \simeq \frac{6667}{16383}, \frac{6668}{16383}$$
$$\frac{3930}{16383}, \frac{3937}{16383} \simeq \frac{6643}{16383}, \frac{6644}{16383}$$
$$\frac{3931}{16383}, \frac{3932}{16383} \simeq \frac{6641}{16383}, \frac{6638}{16383}$$
$$\frac{3933}{16383}, \frac{3934}{16383} \simeq \frac{2221}{5461}, \frac{6664}{16383}$$
$$\frac{3935}{16383}, \frac{3936}{16383} \simeq \frac{2222}{5461}, \frac{2215}{5461}$$
$$\frac{3941}{16383}, \frac{3942}{16383} \simeq \frac{6703}{16383}, \frac{6704}{16383}$$
$$\frac{3943}{16383}, \frac{3976}{16383} \simeq \frac{6610}{16383}, \frac{6605}{16383}$$
$$\frac{3944}{16383}, \frac{3975}{16383} \simeq \frac{6706}{16383}, \frac{6701}{16383}$$
$$\frac{3945}{16383}, \frac{3954}{16383} \simeq \frac{2196}{5461}, \frac{6587}{16383}$$
$$\frac{3946}{16383}, \frac{3953}{16383} \simeq \frac{6595}{16383}, \frac{6596}{16383}$$
$$\frac{3947}{16383}, \frac{3948}{16383} \simeq \frac{6593}{16383}, \frac{6590}{16383}$$
$$\frac{3949}{16383}, \frac{3950}{16383} \simeq \frac{6584}{16383}, \frac{6583}{16383}$$
$$\frac{3951}{16383}, \frac{3952}{16383} \simeq \frac{2199}{5461}, \frac{6586}{16383}$$
$$\frac{3955}{16383}, \frac{3972}{16383} \simeq \frac{6694}{16383}, \frac{6617}{16383}$$
$$\frac{3956}{16383}, \frac{3971}{16383} \simeq \frac{6697}{16383}, \frac{6614}{16383}$$
$$\frac{3957}{16383}, \frac{3958}{16383} \simeq \frac{2229}{5461}, \frac{6688}{16383}$$
$$\frac{3959}{16383}, \frac{3960}{16383} \simeq \frac{2230}{5461}, \frac{6685}{16383}$$
$$\frac{3961}{16383}, \frac{3970}{16383} \simeq \frac{6700}{16383}, \frac{2233}{5461}$$

$$\frac{3962}{16383}, \frac{3969}{16383} \simeq \frac{2204}{5461}, \frac{6611}{16383}$$
$$\frac{3963}{16383}, \frac{3964}{16383} \simeq \frac{2203}{5461}, \frac{2202}{5461}$$
$$\frac{3965}{16383}, \frac{3966}{16383} \simeq \frac{2232}{5461}, \frac{6695}{16383}$$
$$\frac{3967}{16383}, \frac{3968}{16383} \simeq \frac{6698}{16383}, \frac{6613}{16383}$$
$$\frac{3973}{16383}, \frac{3974}{16383} \simeq \frac{6608}{16383}, \frac{6607}{16383}$$
$$\frac{3979}{16383}, \frac{3980}{16383} \simeq \frac{6689}{16383}, \frac{6686}{16383}$$
$$\frac{3981}{16383}, \frac{3982}{16383} \simeq \frac{6616}{16383}, \frac{2205}{5461}$$
$$\frac{3987}{16383}, \frac{4132}{16383} \simeq \frac{6854}{16383}, \frac{6841}{16383}$$
$$\frac{3988}{16383}, \frac{4131}{16383} \simeq \frac{6857}{16383}, \frac{6838}{16383}$$
$$\frac{3989}{16383}, \frac{3990}{16383} \simeq \frac{6847}{16383}, \frac{6848}{16383}$$
$$\frac{3991}{16383}, \frac{3992}{16383} \simeq \frac{6850}{16383}, \frac{6845}{16383}$$
$$\frac{3993}{16383}, \frac{4130}{16383} \simeq \frac{6860}{16383}, \frac{6859}{16383}$$
$$\frac{3994}{16383}, \frac{4129}{16383} \simeq \frac{9524}{16383}, \frac{9523}{16383}$$
$$\frac{3995}{16383}, \frac{3996}{16383} \simeq \frac{9521}{16383}, \frac{9518}{16383}$$
$$\frac{3997}{16383}, \frac{3998}{16383} \simeq \frac{9544}{16383}, \frac{3181}{5461}$$
$$\frac{4000}{16383}, \frac{4127}{16383} \simeq \frac{18}{43}, \frac{25}{43}$$
$$\frac{4001}{16383}, \frac{4122}{16383} \simeq \frac{6836}{16383}, \frac{6835}{16383}$$
$$\frac{4002}{16383}, \frac{4121}{16383} \simeq \frac{9548}{16383}, \frac{9547}{16383}$$
$$\frac{4003}{16383}, \frac{4116}{16383} \simeq \frac{9545}{16383}, \frac{9526}{16383}$$
$$\frac{4004}{16383}, \frac{4115}{16383} \simeq \frac{9542}{16383}, \frac{9529}{16383}$$
$$\frac{4005}{16383}, \frac{4006}{16383} \simeq \frac{9455}{16383}, \frac{3152}{5461}$$
$$\frac{4007}{16383}, \frac{4040}{16383} \simeq \frac{9485}{16383}, \frac{9458}{16383}$$
$$\frac{4008}{16383}, \frac{4039}{16383} \simeq \frac{9490}{16383}, \frac{3151}{5461}$$
$$\frac{4009}{16383}, \frac{4018}{16383} \simeq \frac{9467}{16383}, \frac{3156}{5461}$$
$$\frac{4010}{16383}, \frac{4017}{16383} \simeq \frac{9475}{16383}, \frac{9476}{16383}$$
$$\frac{4011}{16383}, \frac{4012}{16383} \simeq \frac{9470}{16383}, \frac{9473}{16383}$$
$$\frac{4013}{16383}, \frac{4014}{16383} \simeq \frac{9463}{16383}, \frac{9464}{16383}$$
$$\frac{4015}{16383}, \frac{4016}{16383} \simeq \frac{3159}{5461}, \frac{9466}{16383}$$
$$\frac{4019}{16383}, \frac{4036}{16383} \simeq \frac{9497}{16383}, \frac{9446}{16383}$$
$$\frac{4020}{16383}, \frac{4035}{16383} \simeq \frac{9494}{16383}, \frac{9449}{16383}$$
$$\frac{4021}{16383}, \frac{4022}{16383} \simeq \frac{9439}{16383}, \frac{9440}{16383}$$
$$\frac{4023}{16383}, \frac{4024}{16383} \simeq \frac{9442}{16383}, \frac{9437}{16383}$$
$$\frac{4025}{16383}, \frac{4034}{16383} \simeq \frac{9452}{16383}, \frac{9451}{16383}$$
$$\frac{4026}{16383}, \frac{4033}{16383} \simeq \frac{3164}{5461}, \frac{9491}{16383}$$
$$\frac{4027}{16383}, \frac{4028}{16383} \simeq \frac{3163}{5461}, \frac{3162}{5461}$$
$$\frac{4029}{16383}, \frac{4030}{16383} \simeq \frac{9448}{16383}, \frac{3149}{5461}$$

$$\frac{4031}{16383}, \frac{4032}{16383} \simeq \frac{9493}{16383}, \frac{3150}{5461}$$
$$\frac{4037}{16383}, \frac{4038}{16383} \simeq \frac{9487}{16383}, \frac{9488}{16383}$$
$$\frac{4041}{16383}, \frac{4114}{16383} \simeq \frac{3188}{5461}, \frac{9563}{16383}$$
$$\frac{4042}{16383}, \frac{4113}{16383} \simeq \frac{9572}{16383}, \frac{9571}{16383}$$
$$\frac{4043}{16383}, \frac{4044}{16383} \simeq \frac{9569}{16383}, \frac{9566}{16383}$$
$$\frac{4045}{16383}, \frac{4046}{16383} \simeq \frac{6808}{16383}, \frac{2269}{5461}$$
$$\frac{4047}{16383}, \frac{4112}{16383} \simeq \frac{9562}{16383}, \frac{6821}{16383}$$
$$\frac{4048}{16383}, \frac{4111}{16383} \simeq \frac{3191}{5461}, \frac{2270}{5461}$$
$$\frac{4049}{16383}, \frac{4106}{16383} \simeq \frac{6820}{16383}, \frac{2273}{5461}$$
$$\frac{4050}{16383}, \frac{4105}{16383} \simeq \frac{6812}{16383}, \frac{6811}{16383}$$
$$\frac{4051}{16383}, \frac{4068}{16383} \simeq \frac{6790}{16383}, \frac{2259}{5461}$$
$$\frac{4052}{16383}, \frac{4067}{16383} \simeq \frac{6793}{16383}, \frac{2258}{5461}$$
$$\frac{4053}{16383}, \frac{4054}{16383} \simeq \frac{2261}{5461}, \frac{6784}{16383}$$
$$\frac{4055}{16383}, \frac{4056}{16383} \simeq \frac{2262}{5461}, \frac{6781}{16383}$$
$$\frac{4057}{16383}, \frac{4066}{16383} \simeq \frac{6796}{16383}, \frac{2265}{5461}$$
$$\frac{4058}{16383}, \frac{4065}{16383} \simeq \frac{6772}{16383}, \frac{2257}{5461}$$
$$\frac{4059}{16383}, \frac{4060}{16383} \simeq \frac{6769}{16383}, \frac{6766}{16383}$$
$$\frac{4061}{16383}, \frac{4062}{16383} \simeq \frac{2264}{5461}, \frac{6791}{16383}$$

$$\frac{4063}{16383}, \frac{4064}{16383} \simeq \frac{158}{381}, \frac{6773}{16383}$$
$$\frac{4069}{16383}, \frac{4070}{16383} \simeq \frac{6832}{16383}, \frac{2277}{5461}$$
$$\frac{4071}{16383}, \frac{4104}{16383} \simeq \frac{9554}{16383}, \frac{3183}{5461}$$
$$\frac{4072}{16383}, \frac{4103}{16383} \simeq \frac{2278}{5461}, \frac{6829}{16383}$$
$$\frac{4073}{16383}, \frac{4082}{16383} \simeq \frac{9532}{16383}, \frac{3177}{5461}$$
$$\frac{4074}{16383}, \frac{4081}{16383} \simeq \frac{3180}{5461}, \frac{9539}{16383}$$
$$\frac{4075}{16383}, \frac{4076}{16383} \simeq \frac{3179}{5461}, \frac{3178}{5461}$$
$$\frac{4077}{16383}, \frac{4078}{16383} \simeq \frac{3176}{5461}, \frac{9527}{16383}$$
$$\frac{4079}{16383}, \frac{4080}{16383} \simeq \frac{9541}{16383}, \frac{9530}{16383}$$
$$\frac{4083}{16383}, \frac{4100}{16383} \simeq \frac{3187}{5461}, \frac{2274}{5461}$$
$$\frac{4084}{16383}, \frac{4099}{16383} \simeq \frac{3186}{5461}, \frac{2275}{5461}$$
$$\frac{4085}{16383}, \frac{4086}{16383} \simeq \frac{2272}{5461}, \frac{6815}{16383}$$
$$\frac{4087}{16383}, \frac{4088}{16383} \simeq \frac{6818}{16383}, \frac{2271}{5461}$$
$$\frac{4089}{16383}, \frac{4098}{16383} \simeq \frac{2276}{5461}, \frac{6827}{16383}$$
$$\frac{4090}{16383}, \frac{4097}{16383} \simeq \frac{9556}{16383}, \frac{3185}{5461}$$
$$\frac{4091}{16383}, \frac{4092}{16383} \simeq \frac{9553}{16383}, \frac{9550}{16383}$$
$$\frac{4093}{16383}, \frac{4094}{16383} \simeq \frac{6824}{16383}, \frac{6823}{16383}$$
$$\frac{4095}{16383}, \frac{4096}{16383} \simeq \frac{6826}{16383}, \frac{9557}{16383}$$

$$\frac{4101}{16383}, \frac{4102}{16383} \simeq \frac{3184}{5461}, \frac{9551}{16383}$$
$$\frac{4107}{16383}, \frac{4108}{16383} \simeq \frac{6817}{16383}, \frac{6814}{16383}$$
$$\frac{4109}{16383}, \frac{4110}{16383} \simeq \frac{9560}{16383}, \frac{9559}{16383}$$
$$\frac{4117}{16383}, \frac{4118}{16383} \simeq \frac{9535}{16383}, \frac{9536}{16383}$$
$$\frac{4119}{16383}, \frac{4120}{16383} \simeq \frac{9538}{16383}, \frac{9533}{16383}$$
$$\frac{4123}{16383}, \frac{4124}{16383} \simeq \frac{6833}{16383}, \frac{6830}{16383}$$
$$\frac{4125}{16383}, \frac{4126}{16383} \simeq \frac{6856}{16383}, \frac{2285}{5461}$$
$$\frac{4133}{16383}, \frac{4134}{16383} \simeq \frac{6767}{16383}, \frac{2256}{5461}$$
$$\frac{4137}{16383}, \frac{4146}{16383} \simeq \frac{2260}{5461}, \frac{6779}{16383}$$
$$\frac{4138}{16383}, \frac{4145}{16383} \simeq \frac{6787}{16383}, \frac{6788}{16383}$$
$$\frac{4139}{16383}, \frac{4140}{16383} \simeq \frac{6785}{16383}, \frac{6782}{16383}$$
$$\frac{4141}{16383}, \frac{4142}{16383} \simeq \frac{6776}{16383}, \frac{6775}{16383}$$
$$\frac{4143}{16383}, \frac{4144}{16383} \simeq \frac{2263}{5461}, \frac{6778}{16383}$$
$$\frac{4149}{16383}, \frac{4150}{16383} \simeq \frac{3189}{5461}, \frac{9568}{16383}$$
$$\frac{4151}{16383}, \frac{4152}{16383} \simeq \frac{3190}{5461}, \frac{9565}{16383}$$
$$\frac{4155}{16383}, \frac{4156}{16383} \simeq \frac{9617}{16383}, \frac{9614}{16383}$$
$$\frac{4157}{16383}, \frac{4158}{16383} \simeq \frac{6760}{16383}, \frac{2253}{5461}$$
$$\frac{4159}{16383}, \frac{4160}{16383} \simeq \frac{6805}{16383}, \frac{2254}{5461}$$

$$\frac{4165}{16383}, \frac{4166}{16383} \simeq \frac{3205}{5461}, \frac{9616}{16383}$$
$$\frac{4171}{16383}, \frac{4172}{16383} \simeq \frac{3146}{5461}, \frac{3147}{5461}$$
$$\frac{4173}{16383}, \frac{4174}{16383} \simeq \frac{3165}{5461}, \frac{9496}{16383}$$
$$\frac{4175}{16383}, \frac{4176}{16383} \simeq \frac{9509}{16383}, \frac{9434}{16383}$$
$$\frac{4179}{16383}, \frac{4196}{16383} \simeq \frac{3155}{5461}, \frac{9478}{16383}$$
$$\frac{4180}{16383}, \frac{4195}{16383} \simeq \frac{3154}{5461}, \frac{9481}{16383}$$
$$\frac{4181}{16383}, \frac{4182}{16383} \simeq \frac{3157}{5461}, \frac{9472}{16383}$$
$$\frac{4183}{16383}, \frac{4184}{16383} \simeq \frac{9469}{16383}, \frac{3158}{5461}$$
$$\frac{4185}{16383}, \frac{4194}{16383} \simeq \frac{3161}{5461}, \frac{9484}{16383}$$
$$\frac{4186}{16383}, \frac{4193}{16383} \simeq \frac{3153}{5461}, \frac{220}{381}$$
$$\frac{4187}{16383}, \frac{4188}{16383} \simeq \frac{9457}{16383}, \frac{9454}{16383}$$
$$\frac{4189}{16383}, \frac{4190}{16383} \simeq \frac{9479}{16383}, \frac{3160}{5461}$$
$$\frac{4191}{16383}, \frac{4192}{16383} \simeq \frac{9482}{16383}, \frac{9461}{16383}$$
$$\frac{4197}{16383}, \frac{4198}{16383} \simeq \frac{3173}{5461}, \frac{9520}{16383}$$
$$\frac{4201}{16383}, \frac{4210}{16383} \simeq \frac{6844}{16383}, \frac{2281}{5461}$$
$$\frac{4202}{16383}, \frac{4209}{16383} \simeq \frac{6851}{16383}, \frac{2284}{5461}$$
$$\frac{4203}{16383}, \frac{4204}{16383} \simeq \frac{2283}{5461}, \frac{2282}{5461}$$
$$\frac{4205}{16383}, \frac{4206}{16383} \simeq \frac{2280}{5461}, \frac{6839}{16383}$$

$$\frac{4207}{16383}, \frac{4208}{16383} \simeq \frac{6853}{16383}, \frac{6842}{16383}$$
$$\frac{4212}{16383}, \frac{4227}{16383} \simeq \frac{9430}{16383}, \frac{6953}{16383}$$
$$\frac{4213}{16383}, \frac{4214}{16383} \simeq \frac{6943}{16383}, \frac{6944}{16383}$$
$$\frac{4215}{16383}, \frac{4216}{16383} \simeq \frac{6946}{16383}, \frac{6941}{16383}$$
$$\frac{4217}{16383}, \frac{4226}{16383} \simeq \frac{6956}{16383}, \frac{6955}{16383}$$
$$\frac{4218}{16383}, \frac{4225}{16383} \simeq \frac{9428}{16383}, \frac{9427}{16383}$$
$$\frac{4219}{16383}, \frac{4220}{16383} \simeq \frac{9425}{16383}, \frac{9422}{16383}$$
$$\frac{4221}{16383}, \frac{4222}{16383} \simeq \frac{6952}{16383}, \frac{2317}{5461}$$
$$\frac{4223}{16383}, \frac{4224}{16383} \simeq \frac{3143}{5461}, \frac{2318}{5461}$$
$$\frac{4229}{16383}, \frac{4230}{16383} \simeq \frac{3141}{5461}, \frac{9424}{16383}$$
$$\frac{4231}{16383}, \frac{4232}{16383} \simeq \frac{9421}{16383}, \frac{6962}{16383}$$
$$\frac{4235}{16383}, \frac{4236}{16383} \simeq \frac{2314}{5461}, \frac{2315}{5461}$$
$$\frac{4237}{16383}, \frac{4238}{16383} \simeq \frac{6872}{16383}, \frac{6871}{16383}$$
$$\frac{4239}{16383}, \frac{4240}{16383} \simeq \frac{6938}{16383}, \frac{2295}{5461}$$
$$\frac{4245}{16383}, \frac{4246}{16383} \simeq \frac{2197}{5461}, \frac{6592}{16383}$$
$$\frac{4247}{16383}, \frac{4248}{16383} \simeq \frac{2198}{5461}, \frac{6589}{16383}$$
$$\frac{4251}{16383}, \frac{4252}{16383} \simeq \frac{2234}{5461}, \frac{2235}{5461}$$
$$\frac{4253}{16383}, \frac{4254}{16383} \simeq \frac{6727}{16383}, \frac{6728}{16383}$$

$$\frac{4255}{16383}, \frac{4256}{16383} \simeq \frac{6730}{16383}, \frac{6709}{16383}$$
$$\frac{4261}{16383}, \frac{4262}{16383} \simeq \frac{2213}{5461}, \frac{6640}{16383}$$
$$\frac{4263}{16383}, \frac{4264}{16383} \simeq \frac{2223}{5461}, \frac{6674}{16383}$$
$$\frac{4265}{16383}, \frac{4274}{16383} \simeq \frac{2217}{5461}, \frac{6652}{16383}$$
$$\frac{4266}{16383}, \frac{4273}{16383} \simeq \frac{6659}{16383}, \frac{2220}{5461}$$
$$\frac{4267}{16383}, \frac{4268}{16383} \simeq \frac{2218}{5461}, \frac{2219}{5461}$$
$$\frac{4269}{16383}, \frac{4270}{16383} \simeq \frac{6647}{16383}, \frac{2216}{5461}$$
$$\frac{4271}{16383}, \frac{4272}{16383} \simeq \frac{6661}{16383}, \frac{6650}{16383}$$
$$\frac{4275}{16383}, \frac{4292}{16383} \simeq \frac{2227}{5461}, \frac{2210}{5461}$$
$$\frac{4276}{16383}, \frac{4291}{16383} \simeq \frac{2211}{5461}, \frac{2226}{5461}$$
$$\frac{4277}{16383}, \frac{4278}{16383} \simeq \frac{6623}{16383}, \frac{2208}{5461}$$
$$\frac{4279}{16383}, \frac{4280}{16383} \simeq \frac{6626}{16383}, \frac{2207}{5461}$$
$$\frac{4281}{16383}, \frac{4290}{16383} \simeq \frac{2212}{5461}, \frac{6635}{16383}$$
$$\frac{4282}{16383}, \frac{4289}{16383} \simeq \frac{2225}{5461}, \frac{6676}{16383}$$
$$\frac{4283}{16383}, \frac{4284}{16383} \simeq \frac{6673}{16383}, \frac{6670}{16383}$$
$$\frac{4285}{16383}, \frac{4286}{16383} \simeq \frac{6632}{16383}, \frac{6631}{16383}$$
$$\frac{4287}{16383}, \frac{4288}{16383} \simeq \frac{6677}{16383}, \frac{6634}{16383}$$
$$\frac{4293}{16383}, \frac{4294}{16383} \simeq \frac{6671}{16383}, \frac{2224}{5461}$$

$$\frac{4295}{16383},\frac{4296}{16383} \simeq \frac{2214}{5461},\frac{6637}{16383}$$
$$\frac{4299}{16383},\frac{4300}{16383} \simeq \frac{2251}{5461},\frac{2250}{5461}$$
$$\frac{4301}{16383},\frac{4302}{16383} \simeq \frac{9623}{16383},\frac{3208}{5461}$$
$$\frac{4303}{16383},\frac{4304}{16383} \simeq \frac{9637}{16383},\frac{9626}{16383}$$
$$\frac{4305}{16383},\frac{4362}{16383} \simeq \frac{9635}{16383},\frac{3212}{5461}$$
$$\frac{4306}{16383},\frac{4361}{16383} \simeq \frac{3209}{5461},\frac{9628}{16383}$$
$$\frac{4307}{16383},\frac{4324}{16383} \simeq \frac{9593}{16383},\frac{3202}{5461}$$
$$\frac{4308}{16383},\frac{4323}{16383} \simeq \frac{9590}{16383},\frac{3203}{5461}$$
$$\frac{4309}{16383},\frac{4310}{16383} \simeq \frac{9599}{16383},\frac{3200}{5461}$$
$$\frac{4311}{16383},\frac{4312}{16383} \simeq \frac{3199}{5461},\frac{9602}{16383}$$
$$\frac{4313}{16383},\frac{4322}{16383} \simeq \frac{9611}{16383},\frac{3204}{5461}$$
$$\frac{4314}{16383},\frac{4321}{16383} \simeq \frac{9587}{16383},\frac{3196}{5461}$$
$$\frac{4315}{16383},\frac{4316}{16383} \simeq \frac{3195}{5461},\frac{3194}{5461}$$
$$\frac{4317}{16383},\frac{4318}{16383} \simeq \frac{9607}{16383},\frac{9608}{16383}$$
$$\frac{4319}{16383},\frac{4320}{16383} \simeq \frac{9610}{16383},\frac{223}{381}$$
$$\frac{4325}{16383},\frac{4326}{16383} \simeq \frac{9647}{16383},\frac{3216}{5461}$$
$$\frac{4327}{16383},\frac{4328}{16383} \simeq \frac{3270}{5461},\frac{9805}{16383}$$
$$\frac{4329}{16383},\frac{4338}{16383} \simeq \frac{9787}{16383},\frac{9788}{16383}$$
$$\frac{4330}{16383},\frac{4337}{16383} \simeq \frac{3265}{5461},\frac{9796}{16383}$$
$$\frac{4331}{16383},\frac{4332}{16383} \simeq \frac{9790}{16383},\frac{9793}{16383}$$
$$\frac{4333}{16383},\frac{4334}{16383} \simeq \frac{3261}{5461},\frac{9784}{16383}$$
$$\frac{4335}{16383},\frac{4336}{16383} \simeq \frac{9797}{16383},\frac{3262}{5461}$$
$$\frac{4339}{16383},\frac{4356}{16383} \simeq \frac{9817}{16383},\frac{9638}{16383}$$
$$\frac{4340}{16383},\frac{4355}{16383} \simeq \frac{9814}{16383},\frac{9641}{16383}$$
$$\frac{4341}{16383},\frac{4342}{16383} \simeq \frac{9631}{16383},\frac{224}{381}$$
$$\frac{4343}{16383},\frac{4344}{16383} \simeq \frac{9634}{16383},\frac{9629}{16383}$$
$$\frac{4345}{16383},\frac{4354}{16383} \simeq \frac{9644}{16383},\frac{9643}{16383}$$
$$\frac{4346}{16383},\frac{4353}{16383} \simeq \frac{9812}{16383},\frac{9811}{16383}$$
$$\frac{4347}{16383},\frac{4348}{16383} \simeq \frac{9809}{16383},\frac{9806}{16383}$$
$$\frac{4349}{16383},\frac{4350}{16383} \simeq \frac{9640}{16383},\frac{3213}{5461}$$
$$\frac{4351}{16383},\frac{4352}{16383} \simeq \frac{3271}{5461},\frac{3214}{5461}$$
$$\frac{4357}{16383},\frac{4358}{16383} \simeq \frac{3269}{5461},\frac{9808}{16383}$$
$$\frac{4359}{16383},\frac{4360}{16383} \simeq \frac{9650}{16383},\frac{3215}{5461}$$
$$\frac{4363}{16383},\frac{4364}{16383} \simeq \frac{3211}{5461},\frac{3210}{5461}$$
$$\frac{4365}{16383},\frac{4366}{16383} \simeq \frac{9815}{16383},\frac{3272}{5461}$$
$$\frac{4367}{16383},\frac{4368}{16383} \simeq \frac{9829}{16383},\frac{9818}{16383}$$

$$\frac{4371}{16383}, \frac{4372}{16383} \simeq \frac{9785}{16383}, \frac{3266}{5461}$$
$$\frac{4373}{16383}, \frac{4374}{16383} \simeq \frac{9791}{16383}, \frac{3264}{5461}$$
$$\frac{4375}{16383}, \frac{4376}{16383} \simeq \frac{3263}{5461}, \frac{9794}{16383}$$
$$\frac{4378}{16383}, \frac{4385}{16383} \simeq \frac{76}{129}, \frac{3217}{5461}$$
$$\frac{4379}{16383}, \frac{4380}{16383} \simeq \frac{9649}{16383}, \frac{9646}{16383}$$
$$\frac{4381}{16383}, \frac{4382}{16383} \simeq \frac{9671}{16383}, \frac{3224}{5461}$$
$$\frac{4383}{16383}, \frac{4384}{16383} \simeq \frac{9674}{16383}, \frac{9653}{16383}$$
$$\frac{4387}{16383}, \frac{4388}{16383} \simeq \frac{9673}{16383}, \frac{3218}{5461}$$
$$\frac{4389}{16383}, \frac{4390}{16383} \simeq \frac{9071}{16383}, \frac{3024}{5461}$$
$$\frac{4391}{16383}, \frac{4392}{16383} \simeq \frac{9101}{16383}, \frac{9074}{16383}$$
$$\frac{4393}{16383}, \frac{4402}{16383} \simeq \frac{9083}{16383}, \frac{9092}{16383}$$
$$\frac{4394}{16383}, \frac{4401}{16383} \simeq \frac{3028}{5461}, \frac{9091}{16383}$$
$$\frac{4395}{16383}, \frac{4396}{16383} \simeq \frac{9086}{16383}, \frac{9089}{16383}$$
$$\frac{4397}{16383}, \frac{4398}{16383} \simeq \frac{9079}{16383}, \frac{9080}{16383}$$
$$\frac{4399}{16383}, \frac{4400}{16383} \simeq \frac{3031}{5461}, \frac{9082}{16383}$$
$$\frac{4403}{16383}, \frac{4404}{16383} \simeq \frac{9113}{16383}, \frac{3106}{5461}$$
$$\frac{4405}{16383}, \frac{4406}{16383} \simeq \frac{9311}{16383}, \frac{3104}{5461}$$
$$\frac{4407}{16383}, \frac{4408}{16383} \simeq \frac{3103}{5461}, \frac{9314}{16383}$$

$$\frac{4409}{16383}, \frac{4418}{16383} \simeq \frac{9323}{16383}, \frac{3036}{5461}$$
$$\frac{4410}{16383}, \frac{4417}{16383} \simeq \frac{9107}{16383}, \frac{3108}{5461}$$
$$\frac{4411}{16383}, \frac{4412}{16383} \simeq \frac{3035}{5461}, \frac{3034}{5461}$$
$$\frac{4413}{16383}, \frac{4414}{16383} \simeq \frac{9319}{16383}, \frac{9320}{16383}$$
$$\frac{4415}{16383}, \frac{4416}{16383} \simeq \frac{9322}{16383}, \frac{9109}{16383}$$
$$\frac{4419}{16383}, \frac{4420}{16383} \simeq \frac{3107}{5461}, \frac{9110}{16383}$$
$$\frac{4421}{16383}, \frac{4422}{16383} \simeq \frac{9103}{16383}, \frac{9104}{16383}$$
$$\frac{4423}{16383}, \frac{4424}{16383} \simeq \frac{9325}{16383}, \frac{9106}{16383}$$
$$\frac{4425}{16383}, \frac{4498}{16383} \simeq \frac{9179}{16383}, \frac{3084}{5461}$$
$$\frac{4426}{16383}, \frac{4433}{16383} \simeq \frac{3060}{5461}, \frac{9187}{16383}$$
$$\frac{4427}{16383}, \frac{4428}{16383} \simeq \frac{9182}{16383}, \frac{9185}{16383}$$
$$\frac{4429}{16383}, \frac{4430}{16383} \simeq \frac{9239}{16383}, \frac{3080}{5461}$$
$$\frac{4431}{16383}, \frac{4432}{16383} \simeq \frac{3063}{5461}, \frac{9242}{16383}$$
$$\frac{4434}{16383}, \frac{4489}{16383} \simeq \frac{9188}{16383}, \frac{3081}{5461}$$
$$\frac{4435}{16383}, \frac{4436}{16383} \simeq \frac{9209}{16383}, \frac{3074}{5461}$$
$$\frac{4437}{16383}, \frac{4438}{16383} \simeq \frac{3072}{5461}, \frac{9215}{16383}$$
$$\frac{4439}{16383}, \frac{4440}{16383} \simeq \frac{3071}{5461}, \frac{9218}{16383}$$
$$\frac{4441}{16383}, \frac{4450}{16383} \simeq \frac{3068}{5461}, \frac{9227}{16383}$$

$$\frac{4442}{16383},\frac{4449}{16383} \simeq \frac{9203}{16383},\frac{3076}{5461}$$
$$\frac{4443}{16383},\frac{4444}{16383} \simeq \frac{3066}{5461},\frac{3067}{5461}$$
$$\frac{4445}{16383},\frac{4446}{16383} \simeq \frac{9223}{16383},\frac{9224}{16383}$$
$$\frac{4447}{16383},\frac{4448}{16383} \simeq \frac{9205}{16383},\frac{9226}{16383}$$
$$\frac{4451}{16383},\frac{4452}{16383} \simeq \frac{9206}{16383},\frac{3075}{5461}$$
$$\frac{4453}{16383},\frac{4454}{16383} \simeq \frac{9263}{16383},\frac{3088}{5461}$$
$$\frac{4455}{16383},\frac{4456}{16383} \simeq \frac{3055}{5461},\frac{9266}{16383}$$
$$\frac{4457}{16383},\frac{4466}{16383} \simeq \frac{3049}{5461},\frac{3052}{5461}$$
$$\frac{4458}{16383},\frac{4465}{16383} \simeq \frac{9148}{16383},\frac{9155}{16383}$$
$$\frac{4459}{16383},\frac{4460}{16383} \simeq \frac{3050}{5461},\frac{3051}{5461}$$
$$\frac{4461}{16383},\frac{4462}{16383} \simeq \frac{9143}{16383},\frac{24}{43}$$
$$\frac{4463}{16383},\frac{4464}{16383} \simeq \frac{9157}{16383},\frac{9146}{16383}$$
$$\frac{4467}{16383},\frac{4468}{16383} \simeq \frac{3059}{5461},\frac{9254}{16383}$$
$$\frac{4469}{16383},\frac{4470}{16383} \simeq \frac{9247}{16383},\frac{9248}{16383}$$
$$\frac{4471}{16383},\frac{4472}{16383} \simeq \frac{215}{381},\frac{9250}{16383}$$
$$\frac{4473}{16383},\frac{4482}{16383} \simeq \frac{9259}{16383},\frac{9172}{16383}$$
$$\frac{4474}{16383},\frac{4481}{16383} \simeq \frac{3057}{5461},\frac{9260}{16383}$$
$$\frac{4475}{16383},\frac{4476}{16383} \simeq \frac{9169}{16383},\frac{9166}{16383}$$

$$\frac{4477}{16383},\frac{4478}{16383} \simeq \frac{3085}{5461},\frac{9256}{16383}$$
$$\frac{4479}{16383},\frac{4480}{16383} \simeq \frac{3086}{5461},\frac{9173}{16383}$$
$$\frac{4483}{16383},\frac{4484}{16383} \simeq \frac{9257}{16383},\frac{3058}{5461}$$
$$\frac{4485}{16383},\frac{4486}{16383} \simeq \frac{9167}{16383},\frac{3056}{5461}$$
$$\frac{4487}{16383},\frac{4488}{16383} \simeq \frac{3087}{5461},\frac{9170}{16383}$$
$$\frac{4490}{16383},\frac{4497}{16383} \simeq \frac{9244}{16383},\frac{9251}{16383}$$
$$\frac{4491}{16383},\frac{4492}{16383} \simeq \frac{3082}{5461},\frac{3083}{5461}$$
$$\frac{4493}{16383},\frac{4494}{16383} \simeq \frac{9175}{16383},\frac{9176}{16383}$$
$$\frac{4495}{16383},\frac{4496}{16383} \simeq \frac{9253}{16383},\frac{9178}{16383}$$
$$\frac{4499}{16383},\frac{4500}{16383} \simeq \frac{9401}{16383},\frac{3138}{5461}$$
$$\frac{4501}{16383},\frac{4502}{16383} \simeq \frac{9407}{16383},\frac{3136}{5461}$$
$$\frac{4503}{16383},\frac{4504}{16383} \simeq \frac{3135}{5461},\frac{9410}{16383}$$
$$\frac{4505}{16383},\frac{4514}{16383} \simeq \frac{9419}{16383},\frac{3140}{5461}$$
$$\frac{4506}{16383},\frac{4513}{16383} \simeq \frac{2321}{5461},\frac{6964}{16383}$$
$$\frac{4507}{16383},\frac{4508}{16383} \simeq \frac{6961}{16383},\frac{6958}{16383}$$
$$\frac{4509}{16383},\frac{4510}{16383} \simeq \frac{9415}{16383},\frac{9416}{16383}$$
$$\frac{4511}{16383},\frac{4512}{16383} \simeq \frac{9418}{16383},\frac{6965}{16383}$$
$$\frac{4516}{16383},\frac{4643}{16383} \simeq \frac{55}{129},\frac{74}{129}$$

$$\frac{4517}{16383}, \frac{4518}{16383} \simeq \frac{6895}{16383}, \frac{6896}{16383}$$
$$\frac{4519}{16383}, \frac{4520}{16383} \simeq \frac{6925}{16383}, \frac{6898}{16383}$$
$$\frac{4521}{16383}, \frac{4530}{16383} \simeq \frac{6907}{16383}, \frac{6908}{16383}$$
$$\frac{4522}{16383}, \frac{4529}{16383} \simeq \frac{2305}{5461}, \frac{6916}{16383}$$
$$\frac{4523}{16383}, \frac{4524}{16383} \simeq \frac{6910}{16383}, \frac{6913}{16383}$$
$$\frac{4525}{16383}, \frac{4526}{16383} \simeq \frac{2301}{5461}, \frac{6904}{16383}$$
$$\frac{4527}{16383}, \frac{4528}{16383} \simeq \frac{6917}{16383}, \frac{2302}{5461}$$
$$\frac{4531}{16383}, \frac{4532}{16383} \simeq \frac{6937}{16383}, \frac{6886}{16383}$$
$$\frac{4533}{16383}, \frac{4534}{16383} \simeq \frac{2293}{5461}, \frac{160}{381}$$
$$\frac{4535}{16383}, \frac{4536}{16383} \simeq \frac{2294}{5461}, \frac{6877}{16383}$$
$$\frac{4537}{16383}, \frac{4546}{16383} \simeq \frac{6892}{16383}, \frac{2297}{5461}$$
$$\frac{4538}{16383}, \frac{4545}{16383} \simeq \frac{6931}{16383}, \frac{6932}{16383}$$
$$\frac{4539}{16383}, \frac{4540}{16383} \simeq \frac{6929}{16383}, \frac{6926}{16383}$$
$$\frac{4541}{16383}, \frac{4542}{16383} \simeq \frac{2296}{5461}, \frac{6887}{16383}$$
$$\frac{4543}{16383}, \frac{4544}{16383} \simeq \frac{2311}{5461}, \frac{6890}{16383}$$
$$\frac{4547}{16383}, \frac{4548}{16383} \simeq \frac{6934}{16383}, \frac{6889}{16383}$$
$$\frac{4549}{16383}, \frac{4550}{16383} \simeq \frac{2309}{5461}, \frac{6928}{16383}$$
$$\frac{4551}{16383}, \frac{4552}{16383} \simeq \frac{2310}{5461}, \frac{6893}{16383}$$

$$\frac{4553}{16383}, \frac{4626}{16383} \simeq \frac{7004}{16383}, \frac{7003}{16383}$$
$$\frac{4554}{16383}, \frac{4561}{16383} \simeq \frac{2337}{5461}, \frac{7012}{16383}$$
$$\frac{4555}{16383}, \frac{4556}{16383} \simeq \frac{163}{381}, \frac{7006}{16383}$$
$$\frac{4557}{16383}, \frac{4558}{16383} \simeq \frac{9367}{16383}, \frac{9368}{16383}$$
$$\frac{4559}{16383}, \frac{4560}{16383} \simeq \frac{9370}{16383}, \frac{7013}{16383}$$
$$\frac{4562}{16383}, \frac{4617}{16383} \simeq \frac{9371}{16383}, \frac{3124}{5461}$$
$$\frac{4563}{16383}, \frac{4564}{16383} \simeq \frac{9337}{16383}, \frac{9350}{16383}$$
$$\frac{4565}{16383}, \frac{4566}{16383} \simeq \frac{9343}{16383}, \frac{9344}{16383}$$
$$\frac{4567}{16383}, \frac{4568}{16383} \simeq \frac{9341}{16383}, \frac{9346}{16383}$$
$$\frac{4569}{16383}, \frac{4578}{16383} \simeq \frac{9355}{16383}, \frac{9332}{16383}$$
$$\frac{4570}{16383}, \frac{4577}{16383} \simeq \frac{217}{381}, \frac{9356}{16383}$$
$$\frac{4571}{16383}, \frac{4572}{16383} \simeq \frac{9329}{16383}, \frac{9326}{16383}$$
$$\frac{4573}{16383}, \frac{4574}{16383} \simeq \frac{3117}{5461}, \frac{9352}{16383}$$
$$\frac{4575}{16383}, \frac{4576}{16383} \simeq \frac{3118}{5461}, \frac{3111}{5461}$$
$$\frac{4579}{16383}, \frac{4580}{16383} \simeq \frac{9353}{16383}, \frac{9334}{16383}$$
$$\frac{4581}{16383}, \frac{4582}{16383} \simeq \frac{9391}{16383}, \frac{9392}{16383}$$
$$\frac{4583}{16383}, \frac{4584}{16383} \simeq \frac{9394}{16383}, \frac{6989}{16383}$$
$$\frac{4585}{16383}, \frac{4594}{16383} \simeq \frac{2324}{5461}, \frac{6971}{16383}$$

$$\frac{4586}{16383}, \frac{4593}{16383} \simeq \frac{6979}{16383}, \frac{6980}{16383}$$
$$\frac{4587}{16383}, \frac{4588}{16383} \simeq \frac{6977}{16383}, \frac{6974}{16383}$$
$$\frac{4589}{16383}, \frac{4590}{16383} \simeq \frac{6968}{16383}, \frac{6967}{16383}$$
$$\frac{4591}{16383}, \frac{4592}{16383} \simeq \frac{2327}{5461}, \frac{6970}{16383}$$
$$\frac{4595}{16383}, \frac{4596}{16383} \simeq \frac{9382}{16383}, \frac{7001}{16383}$$
$$\frac{4597}{16383}, \frac{4598}{16383} \simeq \frac{3125}{5461}, \frac{9376}{16383}$$
$$\frac{4599}{16383}, \frac{4600}{16383} \simeq \frac{3126}{5461}, \frac{9373}{16383}$$
$$\frac{4601}{16383}, \frac{4610}{16383} \simeq \frac{9388}{16383}, \frac{3129}{5461}$$
$$\frac{4602}{16383}, \frac{4609}{16383} \simeq \frac{2332}{5461}, \frac{6995}{16383}$$
$$\frac{4603}{16383}, \frac{4604}{16383} \simeq \frac{2331}{5461}, \frac{2330}{5461}$$
$$\frac{4605}{16383}, \frac{4606}{16383} \simeq \frac{3128}{5461}, \frac{9383}{16383}$$
$$\frac{4607}{16383}, \frac{4608}{16383} \simeq \frac{9386}{16383}, \frac{6997}{16383}$$
$$\frac{4611}{16383}, \frac{4612}{16383} \simeq \frac{9385}{16383}, \frac{6998}{16383}$$
$$\frac{4613}{16383}, \frac{4614}{16383} \simeq \frac{6992}{16383}, \frac{6991}{16383}$$
$$\frac{4615}{16383}, \frac{4616}{16383} \simeq \frac{9389}{16383}, \frac{6994}{16383}$$
$$\frac{4618}{16383}, \frac{4625}{16383} \simeq \frac{9379}{16383}, \frac{9380}{16383}$$
$$\frac{4619}{16383}, \frac{4620}{16383} \simeq \frac{9377}{16383}, \frac{218}{381}$$
$$\frac{4621}{16383}, \frac{4622}{16383} \simeq \frac{7000}{16383}, \frac{2333}{5461}$$

$$\frac{4623}{16383}, \frac{4624}{16383} \simeq \frac{3127}{5461}, \frac{2334}{5461}$$
$$\frac{4627}{16383}, \frac{4628}{16383} \simeq \frac{6982}{16383}, \frac{2323}{5461}$$
$$\frac{4629}{16383}, \frac{4630}{16383} \simeq \frac{2325}{5461}, \frac{6976}{16383}$$
$$\frac{4631}{16383}, \frac{4632}{16383} \simeq \frac{2326}{5461}, \frac{6973}{16383}$$
$$\frac{4633}{16383}, \frac{4642}{16383} \simeq \frac{6988}{16383}, \frac{2329}{5461}$$
$$\frac{4634}{16383}, \frac{4641}{16383} \simeq \frac{9395}{16383}, \frac{3132}{5461}$$
$$\frac{4635}{16383}, \frac{4636}{16383} \simeq \frac{3131}{5461}, \frac{3130}{5461}$$
$$\frac{4637}{16383}, \frac{4638}{16383} \simeq \frac{2328}{5461}, \frac{6983}{16383}$$
$$\frac{4639}{16383}, \frac{4640}{16383} \simeq \frac{9397}{16383}, \frac{6986}{16383}$$
$$\frac{4645}{16383}, \frac{4646}{16383} \simeq \frac{3109}{5461}, \frac{9328}{16383}$$
$$\frac{4647}{16383}, \frac{4648}{16383} \simeq \frac{3119}{5461}, \frac{3110}{5461}$$
$$\frac{4649}{16383}, \frac{4658}{16383} \simeq \frac{3113}{5461}, \frac{3116}{5461}$$
$$\frac{4650}{16383}, \frac{4657}{16383} \simeq \frac{9340}{16383}, \frac{9347}{16383}$$
$$\frac{4651}{16383}, \frac{4652}{16383} \simeq \frac{3114}{5461}, \frac{3115}{5461}$$
$$\frac{4653}{16383}, \frac{4654}{16383} \simeq \frac{9335}{16383}, \frac{3112}{5461}$$
$$\frac{4655}{16383}, \frac{4656}{16383} \simeq \frac{9349}{16383}, \frac{9338}{16383}$$
$$\frac{4659}{16383}, \frac{4660}{16383} \simeq \frac{3123}{5461}, \frac{2338}{5461}$$
$$\frac{4661}{16383}, \frac{4662}{16383} \simeq \frac{7007}{16383}, \frac{2336}{5461}$$

$$
\begin{aligned}
\frac{4663}{16383}, \frac{4664}{16383} &\simeq \frac{7010}{16383}, \frac{2335}{5461} \\
\frac{4665}{16383}, \frac{4674}{16383} &\simeq \frac{2340}{5461}, \frac{7019}{16383} \\
\frac{4666}{16383}, \frac{4673}{16383} &\simeq \frac{3121}{5461}, \frac{9364}{16383} \\
\frac{4667}{16383}, \frac{4668}{16383} &\simeq \frac{9361}{16383}, \frac{9358}{16383} \\
\frac{4669}{16383}, \frac{4670}{16383} &\simeq \frac{7016}{16383}, \frac{7015}{16383} \\
\frac{4671}{16383}, \frac{4672}{16383} &\simeq \frac{9365}{16383}, \frac{7018}{16383} \\
\frac{4675}{16383}, \frac{4676}{16383} &\simeq \frac{3122}{5461}, \frac{2339}{5461} \\
\frac{4677}{16383}, \frac{4678}{16383} &\simeq \frac{9359}{16383}, \frac{3120}{5461} \\
\frac{4679}{16383}, \frac{4680}{16383} &\simeq \frac{9362}{16383}, \frac{7021}{16383}
\end{aligned}
$$

Bibliography

[DH1] Douady, A. and Hubbard, J. H., *Itération des polynômes quadratiques complexes*, C. R. Acad. Sci. Paris, 294, 1982, p123-126.

[DH2] Douady, A. and Hubbard, J. H., *On the dynamics of polynomial-like mappings*, Ann. sci. de l'École Norm. Sup., 18, 1985, p287-343.

[DH3] Douady, A. and Hubbard, J. H., *A proof of Thurston's topological characterization of rational functions*, Acta Math., 171, 1993, p263-297.

[LA] Lavaurs, P., *Une description combinatoire de l'involution define par M sur les rationnels a denominateur impair*, C. R. Acad. Sci. Paris, 303, 1986, p143-146.

[LE] Lefshetz, S., *On the Fixed Point Formula*, The Annals of Mathematics, Second Series, Vol. 38, No. 4, 1937, p819-822.

[M] Milnor, John, *Dynamics in One Complex Variable, Third Edition*, Princeton University Press, 2006.

[R1] Rees, S. Mary, *Components of degree two hyperbolic rational maps*, Inventiones Math., 100, 1990, p357-382.

[R2] Rees, S. Mary, *A partial description of parameter space of rational maps of degree two: Part 1*, Acta Math., 168, 1992, p11-87.

[R3] Rees, S. Mary, *A partial description of parameter space of rational maps of degree two: Part 2*, Proc. LMS, 70, 1995, p644-690.

[R4] Rees, S. Mary, *Views of Parameter Space: Topographer and Resident*, Société Mathématique de France, 2003.

[R5] Rees, S. Mary, *Multiple equivalent matings with the aeroplane polynomial*, Ergodic Theory and Dynamical Systems, 30, 2010, p1239-1257.

[R6] Rees, S. Mary, *A Fundamental Domain for* V_3, http://arxiv.org/pdf/0904.0328, to appear in Mem. Soc. Math. de France.

[TA] Tan Lei, *Matings of Quadratic Polynomials*, Ergodic Theory and Dynamical Systems, 12, 1992, p589 - 620.

[TH] William P. Thurston, *On the Geometry and Dynamics of Iterated Rational Maps*, Complex dynamics: families and friends, AK Peters, 2008, p3-109.

[W] Ben Wittner, *On the bifurcation loci of rational maps of degree two*, PhD Theses, Cornell University, 1988.